量子計算的數學基礎 MA5501-* Midterm

National Central University, May 11 2022

In this midterm exam, H and X are used to denote the Hadamard gate and the NOT gate, respectively. Moreover, for a given function $f : \{0,1\}^n \to \{0,1\}$, U_f denotes the $(n+1)$ qubit oracle

$$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle\,, \quad \text{where } x \in \{0,1\}^n,\ y \in \{0,1\} \text{ and } \oplus \text{ denotes the addition on } \mathbb{Z}_2\,,$$

and for a give function $f : \{0,1\}^n \to \{0,1\}^n$, Q_f denotes the $2n$ qubit oracle

$$Q_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle\,, \qquad \text{where } x, y \in \{0,1\}^n \text{ and } \oplus \text{ denotes the addition on } \mathbb{Z}_2^n\,.$$

Problem 1. Complete the following.

(1) (10%) Let $\phi_1, \cdots, \phi_n \in \mathbb{R}$. Prove that for each $n \in \mathbb{N}$

$$\begin{aligned}\bigotimes_{\ell=1}^{n}\big(|0\rangle + e^{i\phi_\ell}|1\rangle\big) &\equiv (|0\rangle + e^{i\phi_1}|1\rangle) \otimes (|0\rangle + e^{i\phi_2}|1\rangle) \otimes \cdots \otimes (|0\rangle + e^{i\phi_n}|1\rangle) \\ &= \sum_{j=0}^{2^n-1} e^{i(j_1\phi_1 + j_2\phi_2 + \cdots + j_n\phi_n)}|j\rangle\,,\end{aligned}$$

where $|j\rangle = |j_1 j_2 \cdots j_n\rangle$ if $j = (j_1 j_2 \cdots j_n)_2$.

(2) (5%) Use (1) to deduce that for each $n \in \mathbb{N}$,

$$\mathrm{H}^{\otimes n}|j\rangle \equiv \mathrm{H}^{\otimes n}|j_1 j_2 \cdots j_n\rangle = \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}(-1)^{j\bullet k}|k\rangle\,,$$

where $\bullet$ is the bitwise dot product.

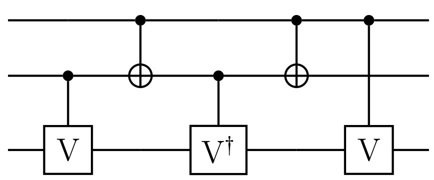
(3) (10%) Write down the definition (using the Dirac $|\cdot\rangle$ notation) of the n-qubit quantum Fourier transform QFT_n (in class it is denoted by F_N with $N = 2^n$). Also show that

$$\mathrm{QFT}_n|0^n\rangle = \mathrm{H}^{\otimes n}|0^n\rangle\,.$$

Problem 2. (10%) Let $f : \{0,1\}^n \to \{0,1\}$. Given the oracle U_f, explain how one implements the quantum gate

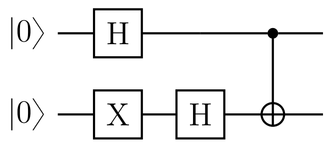
$$|x\rangle \mapsto (-1)^{f(x)}|x\rangle \qquad \text{(in class we use the notation } |i\rangle \mapsto (-1)^{x_i}|i\rangle\text{)}.$$

Problem 3. (15%) Let U, V be 1-qubit quantum gates satisfying $\mathrm{V}^2 = \mathrm{U}$. Show that the quantum circuit



is the same as the controlled-controlled-U gate. Recall that the controlled-controlled-U gate is the 3-qubit quantum gate which activates gate U on the third qubit when the first two qubits are both $|1\rangle$, and otherwise does nothing.

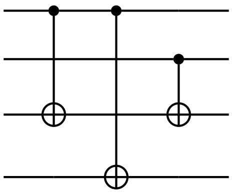
Problem 4. (10%) Show that the final state of the following quantum circuit



is $\frac{1}{2}\big(|00\rangle - |01\rangle - |10\rangle + |11\rangle\big)$ using the matrix representation of the quantum gates/circuits.

Problem 5. Let $f : \{0,1\}^2 \to \{0,1\}^2$ be defined by $f(x_1, x_2) = (x_1 \oplus x_2, x_1)$.

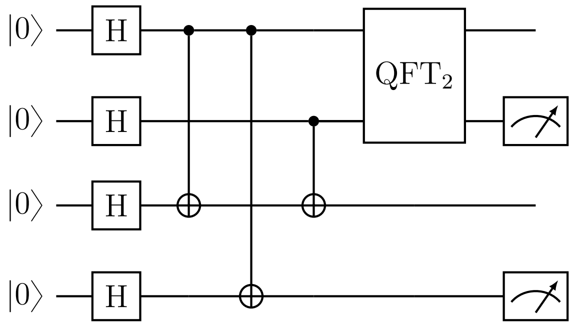
1. (10%) Show that the quantum circuit



   implements Q_f.

2. (5%) Draw the quantum circuit for Simon's algorithm for determining whether f is one-to-one or two-to-one.

3. (15%) Now given the quantum circuit



   and suppose the outcome of the measurement is $|00\rangle$. Find the quantum state after this (projective) measurement.

Problem 6. (10%) Draw the quantum circuite for QFT_3^{-1}, the **inverse** quantum Fourier transform for 3-qubit system, using the Hadamard and the controlled-R_s^* gates, where

$$\mathrm{R}_s\big(\alpha_0|0\rangle + \alpha_1|1\rangle\big) = \alpha_0|0\rangle + \alpha_1 \exp\Big(\frac{2\pi i}{2^s}\Big)|1\rangle\,.$$