國立中央大學數學系

最佳化方法與應用二 MA5038 期中考, Apr. 10, 2024

Problem 1. Consider the constrained optimization problem

$$\min f(x) \qquad \text{subject to} \quad x \in \Omega\,,$$

where the feasible set is given by

$$\Omega = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, c_1(x) \equiv x_1 - x_2^3 \geqslant 0 \text{ and } c_2(x) \equiv x_1 + x_2^3 \geqslant 0\right\}.$$

1. (5%) Justify whether LICQ holds at $(0, 0)$ or not.
2. (5%) Justify whether MFCQ holds at $(0, 0)$ or not.

Problem 2. (25%) Find all the KKT points of the problem

$$\min -x_1x_2 \qquad \text{subject to} \quad \begin{cases} x_1 + x_2^2 \leqslant 2\,, \\ x_1, x_2 \geqslant 0\,. \end{cases}$$

Verify whether these KKT points are local solutions using the second-order sufficient condition if possible.

Problem 3. Consider the functions

$$f(x) = \frac{1}{2}x^{\mathrm{T}}Qx \ - c^{\mathrm{T}}x$$

and

$$f_\mu(x) = \frac{1}{2}x^{\mathrm{T}}Qx \ - c^{\mathrm{T}}x + \mu\phi(x),$$

where $\mu > 0$, $Q \in \mathbb{R}^{n\times n}$ is positive semi-definite, $c \in \mathbb{R}^n$, and $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is given by

$$\phi(x) = \begin{cases} -\sum\limits_{i=1}^{n} \ln x_i & \text{if } x_i > 0,\ x = (x_1, \cdots, x_n), \\ +\infty & \text{otherwise.} \end{cases}$$

1. (15%) Show that ϕ, f and f_μ are convex functions.
2. (15%) Suppose that you are given the fact that for each $\mu > 0$ the solution to the problem $\min f_\mu(x)$ always exists and is unique. Let $\{\mu_k\}$ be a decreasing sequence of positive real scalars with $\mu_k \searrow 0$, and let x^k be the solution to the problem $\min f_{\mu_k}(x)$. Show that if the sequence $\{x^k\}$ has a cluster point $\bar{x}$, then $\bar{x}$ must be a solution to the problem $\min f(x)$ subject to $x \geqslant 0$.

Hint: First show that $\bar{x}$ is a KKT point for the QP $\min f(x)$ subject to $x \geqslant 0$, and then explain why such an $\bar{x}$ is a local solution (thus a global solution).

Problem 4. Consider the following linear programming

$$\min r^{\mathrm{T}}x \quad \text{subject to} \quad Ax \geqslant b\,,$$

where $r \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ are given vectors and matrix. Suppose that the optimal value is attained at some point $x_* \in \mathbb{R}^n$.

1. (15%) Find the dual problem of the linear programming above.

2. (10%) Show that the optimal value of the dual problem is attained at some $\lambda_* \in \mathbb{R}^m$.

3. (10%) Show that the dual problem of the dual problem (minimizing the negative of the objective of the dual problem subject to constraints) is equivalent to the primal problem (that is, the linear programming above).