最佳化方法與應用 MA5038

Homework Assignment 1

Due Apr. 10. 2024

Problem 1. Consider the following constrained optimization problem

$$\min_x (x_1 - 1.5)^2 + (x_2 - t)^4 \quad \text{subject to} \quad \begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geqslant 0,$$

where t be a parameter to be fixed prior to solving the problem. Complete the following.

1. For what values of t does the point $x_* = (1, 0)^{\mathrm{T}}$ satisfy the KKT conditions?
2. Show that when $t = 1$, only the first constraint is active at the solution, and find the solution.

Problem 2. Consider the feasible set Ω in $\mathbb{R}^2$ defined by $x_2 \geqslant 0$, $x_2 \leqslant x_1^2$.

1. For $x_* = (0, 0)^{\mathrm{T}}$, write down $T_\Omega(x_*)$ and $\mathcal{F}(x_*)$.
2. Is LICQ satisfied at x_*? Is MFCQ satisfied?
3. If the objective function is $f(x) = -x_2$, verify that KKT conditions are satisfied at x_*.
4. Find a feasible sequence $\{z_k\}$ approaching x_* with $f(z_k) < f(x_*)$ for all k.

Problem 3. Consider the problem

$$\min_{x \in \mathbb{R}^2} f(x) = -2x_1 + x_2 \quad \text{subject to} \quad \begin{cases} (1 - x_1)^3 - x_2 \geqslant 0, \\ x_2 + 0.25x_1^2 - 1 \geqslant 0. \end{cases}$$

The optimal solution is $x_* = (0, 1)^{\mathrm{T}}$, where both constraints are active.

1. Do the LICQ hold at this point?
2. Are the KKT conditions satisfied?
3. Write down the sets $\mathcal{F}(x_*)$ and $\mathcal{C}(x_*, \lambda_*)$.
4. Are the second-order necessary conditions satisfied? Are the second-order sufficient conditions satisfied?

Problem 4. Consider the constrained optimization problem

$$\min_x f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0 & \text{if } i \in \mathcal{E}, \\ c_i(x) \geqslant 0 & \text{if } i \in \mathcal{I}. \end{cases}$$

Let $\Omega = \big\{x \,\big|\, (\forall\, i \in \mathcal{E})(c_i(x) = 0) \wedge (\forall\, i \in \mathcal{I})(c_i(x) \geqslant 0)\big\}$ be the feasible set. For a point $x \in \Omega$, define the set of KKT multipliers $\mathrm{KKT}(x)$ by

$$\mathrm{KKT}(x) = \left\{ \lambda \in \mathbb{R}^{|\mathcal{E}|+|\mathcal{I}|} \;\middle|\; \begin{array}{c} \nabla_x \mathcal{L}(x, \lambda) = 0 \\ \lambda_i c_i(x) = 0 \text{ if } i \in \mathcal{I} \\ \lambda_i \geqslant 0 \text{ if } i \in \mathcal{I} \end{array} \right\}.$$

Note that in class we "talked" about a characterization for MFCQ:

> Let $x \in \Omega$. Then MFCQ holds at x if and only if the system (for λ)
> $$\sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i \nabla c_i(x) = 0, \quad \lambda_i c_i(x) = 0, \quad i \in \mathcal{I}, \quad \lambda_i \geqslant 0, \quad i \in \mathcal{I}, \tag{$\star$}$$
> only has zero solution.

Use characterization ($\star$) to show that if $\mathrm{KKT}(x) \neq \varnothing$, then

$$\mathrm{KKT}(x) \text{ is compact} \quad \text{if and only if} \quad \text{MFCQ holds at } x$$

by complete the following.

1. For the direction "$\Rightarrow$", assume the contrary that MFCQ does not hold at x. Then characterization ($\star$) of MFCQ provides a non-zero λ; thus for $\mu \in \mathrm{KKT}(x)$, show that $\mu + t\lambda \in \mathrm{KKT}(x)$ for all $t > 0$ and reach a contradiction.

2. For the direction "$\Leftarrow$", first show that $\mathrm{KKT}(x)$ is closed. Then assume the contrary that there exists $\{\lambda_k\} \subseteq \mathrm{KKT}(x)$ such that $\|\lambda_k\| \to \infty$ as $j \to \infty$. Define $\mu_k = \lambda_k/\|\lambda_k\|$, and the Bolzano-Weierstrass theorem implies that there exists a convergent subsequence $\{\mu_{k_j}\}$ with limit $\mu \neq 0$. Show that μ violates characterization ($\star$) of MFCQ.

Problem 5. In this problem you are asked to show ($\star$). Complete the following.

1. Let $x \in \Omega$ be given. Use the dual problem of the following optimization problem

   $$\min_w 0 \quad \text{subject to} \quad \begin{cases} \nabla c_i(x)^{\mathrm{T}} w = 0 & \text{if } i \in \mathcal{E}, \\ \nabla c_i(x)^{\mathrm{T}} w \geqslant 1 & \text{if } i \in \mathcal{A}(x) \cap \mathcal{I}, \end{cases}$$

   to show that $\max_\lambda q(\lambda) = 0$, and use this result to further show

   > There exists $w \in \mathbb{R}^n$ satisfying
   > $\nabla c_i(x)^{\mathrm{T}} w > 0$ for all $i \in \mathcal{A}(x) \cap \mathcal{I}$,
   > $\nabla c_i(x)^{\mathrm{T}} w = 0$ for all $i \in \mathcal{E}$.

   implies that

   > The minimum of the constrained optimization problem
   > $$\max_{\lambda\in\mathbb{R}^{|\mathcal{A}(x)|}} \sum_{i\in\mathcal{A}(x)\cap\mathcal{I}} \lambda_i \quad \text{subject to} \quad \begin{cases} \sum_{i\in\mathcal{A}(x)} \lambda_i \nabla c_i(x) = 0, \\ \lambda_i \geqslant 0, \ i \in \mathcal{A}(x) \cap \mathcal{I}, \end{cases}$$
   > is zero.

   Here $\mathcal{A}(x)$ denotes the active set at x.

2. Use the dual problem of the constrained optimization problem

   $$\min_{\lambda\in\mathbb{R}^{|\mathcal{A}(x)|}} - \sum_{i\in\mathcal{A}(x)\cap\mathcal{I}} \lambda_i \quad \text{subject to} \quad \begin{cases} \sum_{i\in\mathcal{A}(x)} \lambda_i \nabla c_i(x) = 0, \\ \lambda_i \geqslant 0, \ i \in \mathcal{A}(x) \cap \mathcal{I}, \end{cases}$$

to show that

> The minimum of the constrained optimization problem
> $$\max_{\lambda \in \mathbb{R}^{|\mathcal{A}(x)|}} \sum_{i \in \mathcal{A}(x) \cap \mathcal{I}} \lambda_i \quad \text{subject to} \quad \begin{cases} \sum_{i \in \mathcal{A}(x)} \lambda_i \nabla c_i(x) = 0, \\ \lambda_i \geqslant 0, \ i \in \mathcal{A}(x) \cap \mathcal{I}, \end{cases}$$
> is zero.

implies

> There exists $w \in \mathbb{R}^n$ satisfying
> $$\nabla c_i(x)^{\mathrm{T}} w > 0 \text{ for all } i \in \mathcal{A}(x) \cap \mathcal{I},$$
> $$\nabla c_i(x)^{\mathrm{T}} w = 0 \text{ for all } i \in \mathcal{E}.$$

3. Combining part 1 and part 2, we conclude that

> There exists $w \in \mathbb{R}^n$ satisfying
> $$\nabla c_i(x)^{\mathrm{T}} w > 0 \text{ for all } i \in \mathcal{A}(x) \cap \mathcal{I},$$
> $$\nabla c_i(x)^{\mathrm{T}} w = 0 \text{ for all } i \in \mathcal{E}.$$

is equivalent to that

> The minimum of the constrained optimization problem
> $$\max_{\lambda \in \mathbb{R}^{|\mathcal{A}(x)|}} \sum_{i \in \mathcal{A}(x) \cap \mathcal{I}} \lambda_i \quad \text{subject to} \quad \begin{cases} \sum_{i \in \mathcal{A}(x)} \lambda_i \nabla c_i(x) = 0, \\ \lambda_i \geqslant 0, \ i \in \mathcal{A}(x) \cap \mathcal{I}, \end{cases}$$
> is zero.

Use this equivalence to show $(\star)$.