最佳化方法與應用 MA5037

Homework Assignment 1

Due Oct. 11. 2023

Problem 1. Show that Zoutendijk's condition, under certain assumptions on the target function f, also holds when the Wolfe condition is replaced by the strong Wolfe condition: Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous differentiable and bounded from below such that ∇f is Lipschitz. Suppose that in a line search algorithm, at a particular iterate x_k and for a given descent direction p_k, the step length α_k satisfies the Strong Wolfe condition

$$f(x_k + \alpha_k p_k) \leqslant f(x_k) + c_1 \alpha_k \nabla f_k^{\mathrm{T}} p_k \,,$$
$$\left| \nabla f(x_k + \alpha_k p_k)^{\mathrm{T}} p_k \right| \leqslant c_2 \left| \nabla f_k^{\mathrm{T}} p_k \right| ,$$

for some constants c_1, c_2 satisfying $0 < c_1 < c_2 < 1$, then it holds the Zoutendijk condition

$$\sum_{k=0}^{\infty} \cos^2 \theta_k \|\nabla f_k\|^2 < \infty \,,$$

where θ_k is the angle between the descent direction p_k and the steepest descent direction $-\nabla f_k$ satisfying

$$\cos \theta_k = \frac{-\nabla f_k^{\mathrm{T}} p_k}{\|\nabla f_k\| \|p_k\|} \,.$$

Problem 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous differentiable and $\lim\limits_{\|x\| \to \infty} f(x) = \infty$ (so the minimum is inside a bounded region). Consider the steepest descent method with the step length given by the exact line search algorithm; that is, $x_{k+1} = x_k - \alpha_k \nabla f_k$ with the step length α_k given by

$$\alpha_k = \operatorname*{arg\,min}_{\alpha > 0} f(x_k - \alpha \nabla f_k) \,.$$

Show that α_k satisfies both the Wolfe and strong Wolfe conditions if $0 < c_1 \ll 1$ (this is why in practice c_1 is chosen to be 10^{-4}).

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous differentiable. Show that for $0 < c \ll 1$, a step length α obtained from the exact line search satisfies the Goldstein condition.

Problem 4. Consider the steepest descent method with exact line searches applied to the convex quadratic function

$$f(x) = \frac{1}{2} x^{\mathrm{T}} Q x - b^{\mathrm{T}} x \,, \quad \text{Q: positive definite.}$$

Show that if the initial point x_0 satisfies that $x_0 - x_*$, where x_* is the minimizer of f, is an eigenvector of Q, then the steepest descent method will find the solution in one step.