Exercise Problem Sets 11

Jun. 05. 2021

Problem 1. Solve the following PDE using the Laplace transform.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= c^2 \frac{\partial^2 u}{\partial x^2}(x,t) && x > 0, t > 0\,, \\ u(0,t) = f(t)\,,\ &\lim_{x\to\infty} u(x,t) = 0 && t > 0\,, \\ u(x,0) = 0\,,\ &u_t(x,0) = 0 && x > 0\,. \end{aligned}$$

Problem 2. Solve the following PDE using the Laplace transform.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= c^2 \frac{\partial^2 u}{\partial x^2}(x,t) && 0 < x < L, t > 0\,, \\ u(0,t) = 0\,,\ &E u_x(x,t) = F_0 && E \text{ is a constant}\,, t > 0\,, \\ u(x,0) = 0\,,\ &u_t(x,0) = 0 && 0 < x < L\,. \end{aligned}$$

Hint: Expand $1/(1 + e^{-2sL/a})$ in a geometric series.

Problem 3. Solve the following PDE using the Laplace transform.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= c^2 \frac{\partial^2 u}{\partial x^2}(x,t) && x > 0, t > 0\,, \\ u(0,t) = 0\,,\ &\lim_{x\to\infty} u_x(x,t) = 0 && t > 0\,, \\ u(x,0) = 0\,,\ &u_t(x,0) = -\nu_0 && x > 0\,. \end{aligned}$$

Problem 4. Solve the following PDE using the Laplace transform.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) && x > 0, t > 0\,, \\ u(0,t) = 1\,,\ &\lim_{x\to\infty} u(x,t) = 0 && t > 0\,, \\ u(x,0) = e^{-x}\,,\ &u_t(x,0) = 0 && x > 0\,. \end{aligned}$$

Problem 5. Show that a solution to the following PDE

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) + r && x > 0, t > 0\,, \\ u(0,t) = 0\,,\ &\lim_{x\to\infty} u_x(x,t) = 0 && t > 0\,, \\ u(x,0) &= 0 && x > 0\,, \end{aligned}$$

where r is a constant, is given by

$$u(x,t) = rt - r\int_0^t \operatorname{efrc}\Big(\frac{x}{2\alpha\sqrt{\tau}}\Big)\, d\tau\,.$$

Problem 6. Show that a solution to the following PDE

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) - hu(x,t) \qquad && x>0, t>0\,, \\ u(0,t) = 0\,,\ & \lim_{x\to\infty} u_x(x,t) = 0 && t>0\,, \\ u(x,0) &= u_0 && x>0\,, \end{aligned}$$

where h and u_0 are constants, is given by

$$u(x,t) = \frac{u_0 x}{2\sqrt{\pi}} \int_0^t \tau^{-\frac{3}{2}} \exp\Big(-h\tau - \frac{x^2}{4\tau}\Big)\,d\tau\,.$$

Problem 7. Solve the following PDE using the Laplace transform.

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{2}{x}\frac{\partial u}{\partial x}(x,t) \qquad && x>1, t>0\,, \\ u(1,t) = 100\,,\ & \lim_{x\to\infty} u(x,t) = 0 && t>0\,, \\ u(x,0) &= 0 && x>1\,. \end{aligned}$$

Hint: Let $v(x,t) = xu(x,t)$.

Problem 8. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t) - F_0\delta\Big(t - \frac{x}{\nu_0}\Big) \qquad 0<x<L, t>0\,,$$

where $\delta\big(t - x/\nu_0\big)$ is the Dirac delta function. Solve the above PDE subject to

$$\begin{aligned} u(0,t) = 0\,,\ \lim_{x\to\infty} u(x,t) = 0 \qquad & t>0\,, \\ u(x,0) = 0\,,\ u_t(x,0) = 0 \qquad & 0<x<L\,, \end{aligned}$$

when (a) $\nu_0 \neq c$, (b) $\nu_0 = c$.