Exercise Problem Sets 10

May. 24. 2021

Problem 1. Solve the wave equations

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,y) \qquad 0 < x < L\,, t > 0$$

with the following boundary and initial conditions:

1. $u(0,t) = u(L,t) = 0$ for $t > 0$, and $u(x,0) = \frac{1}{4}x(L-x)$, $\frac{\partial u}{\partial t}\Big|_{t=0} = 0$ for $0 < x < L$.
2. $u(0,t) = u(L,t) = 0$ for $t > 0$, and $u(x,0) = 0$, $\frac{\partial u}{\partial t}\Big|_{t=0} = \sin\frac{\pi x}{L}$ for $0 < x < L$.
3. $u(0,t) = u(L,t) = 0$ for $t > 0$, and $u(x,0) = \frac{\partial u}{\partial t}\Big|_{t=0} = x(L-x)$ for $0 < x < L$.

Problem 2. Find a solution of the initial-boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,y) - u(x,y) && 0 < x < \pi\,, t > 0\,, \\ u(0,t) &= u(\pi,t) = 0 && t > 0\,, \\ u(x,0) &= f(x)\,, \frac{\partial u}{\partial t}\Big|_{t=0} = 0 && 0 < x < \pi \end{aligned}$$

where

$$f(x) = \begin{cases} x & \text{if } 0 < x < \pi/2\,, \\ \pi - x & \text{if } \pi \leqslant x < \pi\,. \end{cases}$$

Problem 3. The vertical displacement $u(x,t)$ of an infinitely long string is determined from the initial-value problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad x \in \mathbb{R}\,, t > 0\,, \tag{0.1a}$$

$$u(x,0) = f(x)\,, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = g(x) \quad x \in \mathbb{R}\,. \tag{0.1b}$$

This problem can be solved by the following procedures.

(1) Show that the wave equation (0.1a) can be put into the form $\frac{\partial^2 v}{\partial \eta \partial \xi} = 0$ by means of the substitution $\xi = x + ct$ and $\eta = x - ct$, and $v(\xi,\eta) = u\big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\big)$.

(2) Integrating the partial differential equation given in (1) to find that

$$u(x,t) = F(x+ct) + G(x-ct)$$

for some functions F and G. Use this expression of solution and the initial condition (0.1b) to show that

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(y)\,dy + C$$

and

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(y)\,dy - C\,,$$

where x_0 is arbitrary and C is a constant.

(3) Use the result in (2) to show that

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy\,.$$

Problem 4. Solve the Laplace equations

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \qquad 0 < x < a\,, 0 < y < b\,,$$

with the following boundary conditions:

1. $\frac{\partial u}{\partial x}(0,y) = u(0,y)$, $u(a,y) = 1$ for $0 < y < b$, and $\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) = 0$ for $0 < x < a$.
2. $a = 1$, $b = \pi$, $u(0,y) = \cos y$, $u(1,y) = 1 + \cos 2y$ for $0 < y < \pi$, and $\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,\pi) = 0$ for $0 < x < 1$.

Problem 5. Consider the Laplace equations

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 &\qquad 0 < x < a\,, 0 < y < b\,,\\
\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) = 0 &\qquad 0 < x < a\,,\\
\frac{\partial u}{\partial x}(0,y) = 0\,, \frac{\partial u}{\partial x}(a,y) = g(y) 0\; 0 < y < b\,.
\end{aligned}$$

Explain why a necessary condition for a solution u to exist is that g satisfy

$$\int_0^b g(y)\,dy = 0\,.$$

The condition above is called a compatibility condition.