Exercise Problem Sets 6

Apr. 11. 2021

Problem 1. Show that $(0,0)$ is an asymptotically stable equilibrium of the nonlinear autonomous system

$$\begin{aligned} x' &= \alpha x - \beta y + y^2\,, \\ y' &= \beta x + \alpha y - xy\,, \end{aligned}$$

when $\alpha < 0$ and an unstable equilibrium when $\alpha > 0$.

Hint: Switch to polar coordinates.

Problem 2. Show that the system

$$\begin{aligned} x' &= -\alpha x + xy\,, \\ y' &= 1 - \beta y - x^2\,, \end{aligned}$$

has a unique equilibrium if $\alpha\beta > 1$ and that this equilibrium is stable if $\beta > 0$

Problem 3. 1. Show that the plane autonomous system

$$\begin{aligned} x' &= -x + y - x^3\,, \\ y' &= -x - y + y^2\,, \end{aligned}$$

has two equilibria by sketching the graphs of $-x + y - x^3 = 0$ and $-x - y + y^2 = 0$, Classify the stability of the trivial equilibrium $(0,0)$.

2. Show that the other equilibrium is a saddle point.

Problem 4. 1. Show that $(0,0)$ is the only equilibrium of the plane autonomous system

$$\begin{aligned} x' &= y\,, \\ y' &= -x - \varepsilon\Big(\frac{1}{3}y^3 - y\Big)\,. \end{aligned}$$

2. Show that $(0,0)$ is unstable if $\varepsilon > 0$. When if $(0,0)$ an unstable spiral equilibrium?

3. Show that $(0,0)$ is stable if $\varepsilon < 0$. When if $(0,0)$ a stable spiral equilibrium?

4. Show that $(0,0)$ is a center if $\varepsilon = 0$.

Problem 5. Use the phase-plane method to show that $(0,0)$ is a center of the plane autonomous system

$$\begin{aligned} x' &= y\,, \\ y' &= -2x^3\,. \end{aligned}$$

Problem 6. Use the phase-plane method to show that the solution of the initial-value problem

$$x'' + 2x - x^2 = 0\,, \qquad x(0) = 1\,, x'(0) = 0$$

is periodic.

Problem 7. 1. Find the equilibria of the plane autonomous system

$$\begin{aligned} x' &= 2xy\,, \\ y' &= 1 - x^2 + y^2\,, \end{aligned}$$

and show that the linearization gives no information about the stability of these critical points.

2. Use the phase-plane method to show that the equilibria in 1 are all centers.