Differential Equations MA2042 Final Exam

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Problem 1. (15pts) Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$, then the third order equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0 \tag{0.1}$$

corresponds to the system

$$x_1' = x_2\,, \tag{0.2a}$$

$$x_2' = x_3\,, \tag{0.2b}$$

$$x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3\,. \tag{0.2c}$$

Show that if $\{y_1, y_2, y_3\}$ and $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3\}$ are fundamental sets of equation (0.1) and (0.2), respectively, then $W[y_1, y_2, y_3](t) = c\,\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t)$, where c is a non-zero constant and W and W denote the Wronskian functions given by

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \quad \text{and} \quad \mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t) = \det\big([\boldsymbol{\varphi}_1 \vdots \boldsymbol{\varphi}_2 \vdots \boldsymbol{\varphi}_3]\big).$$

Problem 2. (15pts) Consider the initial value problem

$$y'' + 2y' + 5y = 0\,, \qquad y(0) = 1, \quad y'(0) = 3\,.$$

Let $x_1 = y$ and $x_2 = y'$. For $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$, $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Find the matrix $\boldsymbol{A}$ and solve the initial value problem by finding $\exp(\boldsymbol{A}t)$.

Problem 3. Let $\boldsymbol{P}(t) = \dfrac{1}{t}\begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$ and $\boldsymbol{f}(t) = \begin{bmatrix} t^2 \\ t^4 \end{bmatrix}$.

(a) (15pts) Find the solution $\boldsymbol{\Phi}$ to $\boldsymbol{\Phi}' = \boldsymbol{P}(t)\boldsymbol{\Phi}$ satisfying the initial condition $\boldsymbol{\Phi}(1) = \boldsymbol{I}_2$, where $\boldsymbol{I}_2$ is the 2×2 identity matrix.

(b) (15pts) Find the general solution of the ODE $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{f}(t)$ using the method of variation of parameters.

(c) (15pts) Find the general solution of the ODE $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{f}(t)$ using Theorem 8.28 in the lecture note.

Problem 4. (a) (20pts) Find the Jordan decomposition of the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

(b) (15%) Find the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$.