Differential Equations MA2041-A Sample Midterm Exam 2

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Problem 1. Consider the initial value problem $y' = 1 + y$ with $y(0) = 0$.

1. Find the local truncation error $\tau_k(h)$ for the improved Euler's method.

2. Find the local truncation error $\tau_k(h)$ for the Taylor's method of order 2.

3. Show that the numerical method

$$y_{k+1} = y_k + \frac{h}{6}(1 + y_k)(6 + 3h + h^2)$$

is a third order numerical method; that is, show that the global truncation error $e_k(h)$ satisfies

$$\big|e_k(h)\big| \leqslant Ch^3 \qquad \forall\, 1 \leqslant k \leqslant \frac{T}{h}\,.$$

for some constant $C > 0$.

Problem 2. Consider the initial value problem $y' = \sin(t^2 + y)$ with $y(0) = 0$.

1. Write the improved Euler method in the form

$$y_{k+1} = y_k + h\Phi(h, t_k, y_k)\,.$$

In other words, find the function Φ such that the iterative scheme above is equivalent to the improved Euler method.

2. Show that $\big|\Phi_y(h, t, y)\big| \leqslant \frac{3}{2}$ if $h < 1$ and $t \in [0, 1]$.

3. Show that the local truncation error $\tau_k(h)$ satisfies

$$\big|\tau_k(h)\big| \leqslant 7h^2 \qquad \forall\, h \leqslant \frac{1}{k}\,.$$

Problem 3. Let $\alpha, \beta, \gamma \in \mathbb{R}$ be constants. Use the variation of parameter to find the general solution to the equation

$$y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = e^{\alpha t}\cos(\gamma t)\,.$$

Problem 4. Find the Wronskian (which is unique up to a constant multiple) of two solutions on $(0, \infty)$ to

$$ty'' + (t - 1)y' + 3y = 0\,.$$

Problem 5. Given a solution $\varphi_1(t) = t^{-1/2}\cos t$ to the equation

$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = 0, \qquad t > 0\,,$$

find the solution to the initial value problem

$$t^2y'' + ty' + (t^2 - \frac{1}{4})y = t^{5/2}, \qquad y(1) = y'(1) = 0\,.$$

Problem 6. Consider the equation

$$t^2y'' + ty' + 9y = -\tan(3\log t)\,, \qquad t > 0\,. \tag{$\star$}$$

1. Let $z(x) = y(e^x)$. Find the ODE that z satisfies.
2. Find the general solution to $(\star)$ by solving for z first.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the boundary value problem

$$y'' + y = f(t)\,, \qquad y(0) = 0,\ y(\pi) = 0$$

has a solution if and only if $\displaystyle\int_0^\pi f(t)\sin t\,dt = 0$.