Differential Equations MA2041-A Midterm Exam 2

National Central University, Dec. 14 2016

學號：____________________ 姓名：____________________

1(1)(2)	1(2)	2(1)	2(2)	2(3)	3	4	5	6	7	Total

Problem 1. Consider the initial value problem $y' = t + y$ with $y(0) = 0$.

1. (5%) Find the exact solution to the initial value problem above.
2. (15%) Show that the numerical method

$$y_{k+1} = y_k + h\big[t_k + \frac{h}{2} + y_k + \frac{h}{2}(t_k + y_k)\big]$$

is a third order numerical method; that is, show that the global truncation error $e_k(h)$ satisfies

$$\big|e_k(h)\big| \leqslant Ch^2 \qquad \forall\, 1 \leqslant k \leqslant \frac{T}{h}\,.$$

for some constant $C > 0$.

Solution:

1. By the method of integrating factors, $(e^{-t}y)' = te^{-t}$ which implies that

$$e^{-t}y = \int te^{-t}\,dt = -te^{-t} + \int e^{-t}\,dt = -(t+1)e^{-t} + C\,.$$

Therefore, $y(t) = Ce^t - t - 1$. Together with the initial condition $y(0) = 0$, we find that $C = 1$; thus

$$y(t) = e^t - t - 1\,.$$

2. By the fact that $e^h = 1 + h + \dfrac{h^2}{2} + \dfrac{e^\theta}{6}h^3$ for some $\theta \in (0, h)$, we find that

$$\begin{aligned}
\tau_k(h) &= \frac{y(t_k) - y(t_{k-1})}{h} - t_{k-1} - \frac{h}{2} - y(t_{k-1}) - \frac{h}{2}\big(t_{k-1} + y(t_{k-1})\big) \\
&= \frac{e^{t_{k-1}}(e^h - 1) - h}{h} - t_{k-1} - \frac{h}{2} - e^{t_{k-1}} + t_{k-1} + 1 - \frac{h}{2}(e^{t_{k-1}} - 1) \\
&= \frac{e^{t_{k-1}}(h + \frac{h^2}{2} + \frac{e^\theta}{6}h^3)}{h} - e^{t_{k-1}} - \frac{h}{2}e^{t_{k-1}} = \frac{e^{t_{k-1}+\theta}}{6}h^2\,.
\end{aligned}$$

Therefore, for $t_k \in [0, T]$, we have

$$\big|\tau_k(h)\big| \leqslant \frac{e^T}{6}h^2\,.$$

Moreover, with Φ denoting the function

$$\Phi(h, t, y) = t + \frac{h}{2} + y + \frac{h}{2}(t + y)\,,$$

the numerical scheme can be expressed as $y_{k+1} = y_k + h\Phi(h, t_k, y_k)$ and $\Phi_y(h, t, y) = 1 + \dfrac{h}{2}$ which further implies that

$$\big|\Phi_y(h, t, y)\big| \leqslant 1 + \frac{T}{2}\,.$$

Therefore, Theorem 3.8 in the lecture note implies that $\big|e_k(h)\big| \leqslant Ch^2$ for some $C > 0$. □

Problem 2. Consider the initial value problem $y' = \cos(t^3 + y)$ with $y(0) = 0$.

1. (5%) Write the improved Euler's method in the form

$$y_{k+1} = y_k + h\Phi(h, t_k, y_k)\,.$$

In other words, find the function Φ such that the iterative scheme above is equivalent to the improved Euler method.

2. (15%) Show that the local truncation error $\tau_k(h)$ satisfies

$$\big|\tau_k(h)\big| \leqslant 9h^2 \qquad \forall\, h \leqslant \frac{1}{k}\,.$$

Proof. First we compute the derivative of y as follows:

$$\begin{aligned} y'' &= -\sin(t^3+y)(3t^2+y') = -\sin(t^3+y)\big(3t^2+\cos(t^3+y)\big)\,,\\ y''' &= -\cos(t^3+y)\big(3t^2+\cos(t^3+y)\big)^2 - \sin(t^3+y)\big[6t - \sin(t^3+y)\big(3t^2+\cos(t^3+y)\big)\big]\,; \end{aligned}$$

thus for $t \in [0,1]$, $\big|y'''(t)\big| \leqslant (3+1)^2 + (6+4) = 26$.

1. Let $f(t,y) = \cos(t^3+y)$. The improved Euler's method is the numerical scheme given by

$$\begin{aligned} y_{k+1} &= y_k + \frac{h}{2}\big[f(t_k,y_k) + f(t_k+h, y_k + hf(t_k,y_k))\big]\\ &= y_k + \frac{h}{2}\big[\cos(t_k^3+y_k) + \cos\big((t_k+h)^3 + y_k + h\cos(t_k^2+y_k)\big)\big]\\ &= y_k + \frac{h}{2}\big[\cos(t_k^3+y_k) + \cos\big(t_k^3 + y_k + 3h^2t_k + 3ht_k^2 + h^3 + h\cos(t_k^3+y_k)\big)\big]\,; \end{aligned}$$

thus $\Phi(h,t,y) = \dfrac{1}{2}\big[\cos(t^3+y) + \cos(t^3+y+3h^2t+3ht^2+h^3+h\cos(t^3+y))\big]$.

2. By the Taylor theorem,

$$\begin{aligned} y(t_k) &= y(t_{k-1}) + hy'(t_{k-1}) + \frac{h^2}{2}y''(t_{k-1}) + \frac{h^3}{6}y'''(\xi_{k-1})\\ &= y(t_{k-1}) + h\cos\big(t_{k-1}^3 + y(t_{k-1})\big) + \frac{h^2}{2}\Big[-3t_{k-1}^2\sin(t_{k-1}^3+y(t_{k-1}))\\ &\qquad - \sin(t_{k-1}^3+y(t_{k-1}))\cos(t_{k-1}^3+y(t_{k-1}))\Big] + \frac{h^3}{6}y'''(\xi_{k-1}) \end{aligned}$$

for some ξ_{k-1} in between t_{k-1} and t_k. Moreover, by the Taylor theorem,

$$\cos x = \cos a - \sin a(x-a) - \frac{\cos a}{2}(x-a)^2 + \frac{\sin b}{2}(x-a)^3$$

for some b in between x and a; thus

$$\begin{aligned} y_k &= y_{k-1} + h\Phi(h,t_{k-1},y_{k-1})\\ &= y_{k-1} + \frac{h}{2}\Big[2\cos(t_{k-1}^3+y_{k-1}) - \sin(t_{k-1}^3+y_{k-1})\big(3t_{k-1}^2h + 3t_{k-1}h^2 + h^3 + h\cos(t_{k-1}^3+y_{k-1})\big)\\ &\qquad - \frac{1}{2}\cos\eta_{k-1}\big(3t_{k-1}^2h + 3t_{k-1}h^2 + h^3 + h\cos(t_{k-1}^3+y_{k-1})\big)^2\Big] \end{aligned}$$

for some η_{k-1} in between $t_{k-1}^3 + y_{k-1}$ and $(t_{k-1}+h)^3 + y_{k-1} + h\cos(t_{k-1}^3 + y_{k-1})$. Therefore, in the time interval $[0,1]$ the local truncation error $\tau_k(h)$ satisfies

$$\begin{aligned}
\big|\tau_k(h)\big| &\leqslant \frac{1}{h}\Big[\frac{h}{2}\big|\sin(t_{k-1}^3 + y_{k-1})\big|\big|3t_{k-1}h^2 + h^3\big| \\
&\qquad + \frac{h^3}{4}\big|\cos\eta_{k-1}\big|\big|3t_{k-1}^2 + 3t_{k-1}h + h^2 + \cos(t_{k-1}^3 + y_{k-1})\big|^2 + \frac{h^3}{6}\big|y'''(\xi_{k-1})\big|\Big] \\
&\leqslant h^2\big[\frac{4}{2} + \frac{1}{4}(3+3+1+1) + \frac{26}{6}\big] = h^3\big(4 + \frac{13}{3}\big) = \frac{25}{3}h^3
\end{aligned}$$

which further implies that $\big|\tau_k(h)\big| \leqslant 9h^2$. □

Problem 3. (15%) Let $\alpha \in \mathbb{R}$ be constants. Use the variation of parameter to find the general solution to the equation

$$y'' - 2\alpha y' + \alpha^2 y = te^{\alpha t}.$$

Solution: First we find a fundamental set of the corresponding homogeneous equation. Since the characteristic equation (of the corresponding homogeneous equation) is $\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$ has double root $\lambda = \alpha$, a fundamental set $\{\varphi_1, \varphi_2\}$ of the corresponding homogeneous equation is given by

$$\varphi_1(t) = e^{\alpha t} \qquad \text{and} \qquad \varphi_2(t) = te^{\alpha t}.$$

Since the Wronskian of φ_1 and φ_2 is $W[\varphi_1, \varphi_2](t) = \begin{vmatrix} e^{\alpha t} & te^{\alpha t} \\ \alpha e^{\alpha t} & e^{\alpha t} + \alpha te^{\alpha t} \end{vmatrix} = e^{2\alpha t}$, using the method of variation of parameters (or formula (4.25) in the lecture note) we find that a particular solution to the non-homogeneous equation is given by

$$y_p(t) = -e^{\alpha t} \int_0^t \frac{se^{\alpha s} se^{\alpha s}}{e^{2\alpha s}}\, ds + te^{-\alpha t} \int_0^t \frac{se^{\alpha s} e^{\alpha s}}{e^{2\alpha s}}\, ds = -\frac{t^3}{3} e^{\alpha t} + \frac{t^3}{2} e^{\alpha t} = \frac{t^3}{6} e^{\alpha t}.$$

Therefore, the general solution to the origin non-homogeneous equation is given by

$$y(t) = C_1 e^{\alpha t} + C_2 te^{\alpha t} + \frac{t^3}{6} e^{\alpha t}.$$

Problem 4. (10%) Find the Wronskian (which is unique up to a constant multiple) of two solutions on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to

$$(\cos t)y'' + (\sin t)y' - ty = 0\,.$$

Solution: Let $\{\varphi_1, \varphi_2\}$ be a collection of two solutions of the ODE, and $W(t) \equiv \begin{vmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(2) \end{vmatrix}$ be the Wronskian of φ_1 and φ_2. Then the Abel theorem implies that

$$W'(t) + \frac{\sin t}{\cos t} W(t) = 0 \qquad \forall\, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Using the separation of variables, we have

$$\frac{dW}{W} = -\tan t dt\,;$$

thus $\log |W(t)| = -\log|\sec t| + C = \log|\cos t| + C$ which further implies that

$$W(t) = C \cos t$$

for some constant C.

Problem 5. (20%) Given a solution $\varphi_1(t) = t^2$ to the equation

$$t^2 y'' - 3ty' + 4y = 0, \qquad t > 0\,,$$

find the solution to the initial value problem

$$t^2 y'' - 3ty' + 4y = t^2 \log t, \qquad y(1) = y'(1) = 0\,,$$

where $\log t = \ln t = \log_e t$.

Solution: First we rewrite the initial value problem as

$$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = \log t, \qquad y(1) = y'(1) = 0\,.$$

Using formula (4.20) in the lecture note, we find that a (particular) solution of the non-homogeneous equation

$$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = \log t$$

can be expressed by

$$y(t) = t^2 \int \frac{\displaystyle\int t^2 e^{-3\log t} \log t\, dt}{t^4 e^{-3\log t}}\, dt = t^2 \Big[\int \frac{1}{t}\Big(\int \frac{\log t}{t}\, dt\Big)\, dt\Big]\,.$$

Since $\displaystyle\int \frac{\log t}{t}\, dt = \frac{1}{2}(\log t)^2 + C$ and $\displaystyle\int \frac{(\log t)^2}{t}\, dt = \frac{1}{3}(\log t)^3 + D$, we find that

$$\int \frac{1}{t}\Big(\int \frac{\log t}{t}\, dt\Big)\, dt = \int \frac{1}{t}\Big(\frac{1}{2}(\log t)^2 + C\Big)\, dt = C\log t + \frac{1}{6}(\log t)^3 + \frac{D}{2}\,;$$

thus the general solution to the non-homogeneous equation is

$$y(t) = C_1 t^2 \log t + C_2 t^2 + \frac{t^2}{6}(\log t)^3\,.$$

To validate the initial condition, C_1 and C_2 must satisfy

$$\begin{aligned} 0 &= y(1) = C_2\,, \\ 0 &= y'(1) = C_1 + 2C_2\,; \end{aligned}$$

thus $C_1 = C_2 = 0$. Therefore, the solution to the original initial value problem is $y(t) = \dfrac{t^2}{6}(\log t)^3$. □

Problem 6. Solve the differential equation

$$\frac{\sin^2(2x)}{4}y''(x) + \sin(2x)\cos^2 x y'(x) - 2y(x) = 0\,, \qquad 0 < x < \frac{\pi}{2} \tag{$\star$}$$

following the steps below:

(1) (10%) Let $t = \tan x$ and $z(t) = y(\arctan t)$. Find the corresponding differential equation that z satisfies (the function arctan is identical to $\tan^{-1}$).

(2) (10%) Find the general solution to the equation for z, and then use it to find a solution to ($\star$).

Solution:

(1) Let $t = \tan x$ and $z(t) = y(\tan^{-1} t)$. Then

$$z'(t) = y'(\tan^{-1} t)\frac{1}{1+t^2} \quad \text{and} \quad z''(t) = y''(\tan^{-1} t)\frac{1}{(1+t^2)^2} + y'(\tan^{-1} t)\frac{-2t}{(1+t^2)^2}\,.$$

Therefore,

$$y'(\tan^{-1} t) = (1+t^2)z'(t) \quad \text{and} \quad y''(\tan^{-1} t) = (1+t^2)^2 z''(t) + 2t(1+t^2)z'(t)\,.$$

Letting $x = \tan^{-1} t$ in the ODE we find that

$$\frac{t^2}{(1+t^2)^2}\,y''(\tan^{-1} t) + \frac{2t}{(1+t^2)^2}y'(\tan^{-1} t) - 2y(\tan^{-1} t) = 0\,;$$

thus

$$t^2 z''(t) + 2tz'(t) - 2z(t) = 0\,.$$

(2) Let r satisfy $r(r-1) + 2r - 2 = 0$. Then $r^2 + r - 2 = 0$ which implies $r = -2$ and $r = 1$. Therefore, the general solution of ($\star$) is

$$z(t) = C_1 t^{-2} + C_2 t\,.$$

Therefore,

$$y(x) = z(\tan x) = C_1 \cot^2 x + C_2 \tan x\,.$$

Problem 7. (15%) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the boundary value problem

$$y'' + 2y' + 2y = f(t)\,, \qquad y(0) = 0,\ y(\pi) = 0 \tag{$\star\star$}$$

has a solution if and only if $\displaystyle\int_0^\pi e^t f(t) \sin t\, dt = 0$.

Proof. First we note that the solution to the corresponding homogeneous equation

$$y'' + 2y' + 2y = 0$$

is $y(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t$ since $-1 \pm i$ are the roots of the characteristic equation $\lambda^2 + 2\lambda + 2 = 0$. Let $\varphi_1(t) = e^{-t}\cos t$ and $\varphi_2(t) = e^{-t}\sin t$. Then the Wronskian of φ_1 and φ_2 is

$$W[\varphi_1, \varphi_2](t) = \begin{vmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \end{vmatrix} = \begin{vmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{vmatrix} = e^{-2t}\,.$$

Using the method of variation of parameters (or formula (4.25) in the lecture note), we find that a particular solution can be written as

$$y_p(t) = -e^{-t}\cos t \int_0^t \frac{f(s)e^{-s}\sin s}{e^{-2s}}\, ds + e^{-t}\sin t \int_0^t \frac{f(s)e^{-s}\cos s}{e^{-2s}}\, ds\,;$$

thus the general solution to the non-homogeneous ODE

$$y'' + 2y' + 2y = f(t)$$

is given by

$$\begin{aligned} y(t) &= C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + y_p(t) \\ &= C_1 e^{-t}\cos t + C_2 e^{-t}\sin t - e^{-t}\cos t \int_0^t f(s)e^s \sin s\, ds + e^{-t}\sin t \int_0^t f(s)e^s\cos s\, ds\,. \end{aligned}$$

Therefore, the boundary value problem ($\star\star$) has a solution if and only if

$$0 = y(0) = C_1$$

and

$$0 = y(\pi) = -C_1 e^{-\pi} + e^{-\pi}\int_0^\pi f(s)e^s \sin s\, ds\,.$$

Therefore, the boundary value problem ($\star\star$) has a solution if and only if

$$\int_0^\pi f(s)e^s\sin s\, ds = 0 \tag{$\star\star\star$}$$

and the solution, provided that ($\star\star\star$) holds, is given by

$$y(t) = C_2 e^{-t}\sin t - e^{-t}\cos t \int_0^t f(s)e^s\sin s\, ds + e^{-t}\sin t\int_0^t f(s)e^s\cos s\, ds\,. \qquad \square$$