Differential Equations MA2041-A Final Exam

National Central University, Jan. 11 2017

學號：____________________ 姓名：____________________

Problem 1. Complete the following.

1. (10pts) Given a solution $y = \varphi_1(t) = t^2$ to
$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0 \qquad \text{for} \quad t > 0\,,$$
find a fundamental set of the ODE above.

2. (15pts) Find the general solution to the inhomogeneous ODE
$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = t^2(t+3)^2 \qquad \text{for} \quad t > 0\,.$$

Solution:

1. Suppose that $y = v(t)t^2$ is also a solution to the ODE above. Then
$$t^2(t+3)(v'''t^2 + 6v''t + 6v') - 3t(t+2)(v''t^2 + 4v't + 2v) + 6(1+t)(v't^2 + 2vt) - 6vt^2 = 0$$
which implies that v satisfies
$$t^4(t+3)v''' + \big[6t^3(t+3) - 3t^3(t+2)\big]v'' + \big[6t^2(t+3) - 12t^2(t+2) + 6t^2(1+t)\big]v' = 0$$
or equivalently, with u denoting v'',
$$t(t+3)u' + 3(t+4)u = 0\,.$$
Solving the ODE above, we find that $u(t) = C_1 t^{-4}(t+3)$ for some constant C_1; thus
$$v(t) = \frac{C_1}{2}(t^{-2} + t^{-1}) + C_2 t + C_3$$
for some constants C_2 and C_2. Therefore, the general solution to the ODE above is given by $y(t) = C_1(1+t) + C_2t^3 + C_3t^2$ which implies that $\{t^2, t^3, 1+t\}$ is a fundamental set of the ODE.

2. Let $\varphi_1(t) = t^2$, $\varphi_2(t) = t^3$ and $\varphi_3(t) = t+1$. Then
$$W[\varphi_1, \varphi_2, \varphi_3](t) = \begin{vmatrix} t^2 & t^3 & 1+t \\ 2t & 3t^2 & 1 \\ 2 & 6t & 0 \end{vmatrix} = 12t^2(1+t) + 2t^3 - 6t^2(1+t) - 6t^3 = 2t^2(t+3)\,.$$
and
$$W_1(t) = \begin{vmatrix} t^3 & 1+t \\ 3t^2 & 1 \end{vmatrix} = -2t^3 - 3t^2\,,\ W_2(t) = \begin{vmatrix} t^2 & 1+t \\ 2t & 1 \end{vmatrix} = -t^2 - 2t\,,\ W_3(t) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^4\,.$$

Rewrite the initial value problem as

$$y''' - \frac{3t(t+2)}{t^2(t+3)}y'' + \frac{6(1+t)}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = (t+3)\,.$$

Let $g(t) = t+3$. Using the formula in the lecture note, we find that the general solution to the ODE is given by

$$\begin{aligned} y(t) &= -\varphi_1(t)\int \frac{(2t^3+3t^2)g(t)}{2t^2(t+3)}\,dt + \varphi_2(t)\int \frac{(t^2+2t)g(t)}{2t^2(t+3)}\,dt + \varphi_3(t)\int \frac{t^4 g(t)}{2t^2(t+3)}\,dt \\ &= -\varphi_1(t)\big(\frac{1}{2}t^2+\frac{3}{2}t\big) + \varphi_2(t)\big(\frac{t}{2}+\ln t\big) + \varphi_3(t)\frac{t^3}{6} + C_1\varphi_1(t) + C_2\varphi_2(t) + C_3\varphi_3(t) \\ &= C_1\varphi_1(t) + C_2\varphi_2(t) + C_3\varphi_3(t) + \frac{1}{6}t^4 - \frac{4}{3}t^3 + t^3\ln t\,. \end{aligned}$$

□

Problem 2. Solve the initial value problem

$$y'' - 4y' + 5y = 5t^2 - 8t + 7\,, \qquad y(0) = 0\,, \quad y'(0) = 1$$

using

1. (20pts) the method of variation of parameters.
2. (20pts) the method of annihilator.
3. (20pts) the Laplace transform.

Solution:

1. The zeros of the characteristic equation of the corresponding homogeneous equation is $r = 2 \pm i$. Let $\varphi_1(t) = e^{2t}\cos t$, $\varphi_2(t) = e^{2t}\sin t$ and $g(t) = 5t^2 - 8t + 7$. Then

$$W[\varphi_1,\varphi_2](t) = \begin{vmatrix} e^{2t}\cos t & e^{2t}\sin t \\ 2e^{2t}\cos t - e^{2t}\sin t & 2e^{2t}\sin t + e^{2t}\cos t \end{vmatrix} = e^{4t}$$

and $W_1(t) = e^{2t}\sin t$, $W_2(t) = e^{2t}\cos t$. Using formula in the lecture note, we find that the general solution is given by

$$\begin{aligned} y(t) &= -\varphi_1(t)\int \frac{e^{2t}\sin t(5t^2-8t+7)}{e^{4t}}\,dt + \varphi_2(t)\int \frac{e^{2t}\cos t(5t^2-8t+7)}{e^{4t}}\,dt \\ &= -\varphi_1(t)\int e^{-2t}\sin t(5t^2-8t+7)\,dt + \varphi_2(t)\int e^{-2t}\cos t(5t^2-8t+7)\,dt\,. \end{aligned}$$

Since

$$\begin{aligned} \int e^{-2t}\cos t\,dt &= e^{-2t}\sin t + 2\int e^{-2t}\sin t\,dt = e^{-2t}\sin t - 2\Big[e^{-2t}\cos t + 2\int e^{-2t}\sin t\,dt\Big] \\ &= e^{-2t}(\sin t - 2\cos t) - 4\int e^{-2t}\sin t\,dt\,, \end{aligned}$$

we find that

$$\int e^{-2t}\cos t\,dt = \frac{e^{-2t}}{5}(\sin t - 2\cos t) + C$$

and the identity above further implies that

$$\int e^{-2t}\sin t\,dt = -\frac{e^{-2t}}{5}(2\sin t+\cos t)+C\,.$$

Integrating by parts,

$$\begin{aligned}
&\int e^{-2t}\sin t(5t^2-8t+7)\,dt\\
&\quad= -\frac{e^{-2t}}{5}(2\sin t+\cos t)(5t^2-8t+7)+\int \frac{e^{-2t}}{5}(2\sin t+\cos t)(10t-8)\,dt\\
&\quad= -\frac{e^{-2t}}{5}(2\sin t+\cos t)(5t^2-8t+7)-\frac{e^{-2t}}{25}(3\sin t+4\cos t)(10t-8)\\
&\qquad+10\int \frac{e^{-2t}}{25}(3\sin t+4\cos t)\\
&\quad= -\frac{e^{-2t}}{5}(2\sin t+\cos t)(5t^2-8t+7)-\frac{e^{-2t}}{25}(3\sin t+4\cos t)(10t-8)\\
&\qquad-\frac{2e^{-2t}}{25}(2\sin t+11\cos t)+C_1
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\int e^{-2t}\cos t(5t^2-8t+7)\,dt\\
&\quad= \frac{e^{-2t}}{5}(\sin t-2\cos t)(5t^2-8t+7)-\int \frac{e^{-2t}}{5}(\sin t-2\cos t)(10t-8)\,dt\\
&\quad= \frac{e^{-2t}}{5}(\sin t-2\cos t)(5t^2-8t+7)+\frac{e^{-2t}}{25}(4\sin t-3\cos t)(10t-8)\\
&\qquad-10\int \frac{e^{-2t}}{25}(4\sin t-3\cos t)\,dt\\
&\quad= \frac{e^{-2t}}{5}(\sin t-2\cos t)(5t^2-8t+7)+\frac{e^{-2t}}{25}(4\sin t-3\cos t)(10t-8)\\
&\qquad+\frac{2e^{-2t}}{25}(11\sin t-2\cos t)+C_2\,.
\end{aligned}$$

Therefore, the general solution to the ODE is given by

$$\begin{aligned}
y(t) &= -\cos t\Big[-\frac{1}{5}(2\sin t+\cos t)(5t^2-8t+7)-\frac{1}{25}(3\sin t+4\cos t)(10t-8)\\
&\qquad\qquad-\frac{2}{25}(2\sin t+11\cos t)\Big]\\
&\quad+\sin t\Big[\frac{1}{5}(\sin t-2\cos t)(5t^2-8t+7)+\frac{1}{25}(4\sin t-3\cos t)(10t-8)\\
&\qquad\qquad+\frac{2}{25}(11\sin t-2\cos t)\Big]\\
&\quad+C_1e^{2t}\cos t+C_2e^{2t}\sin t\\
&= \frac{1}{5}(5t^2-8t+7)+\frac{4}{25}(10t-8)+\frac{22}{25}+C_1e^{2t}\cos t+C_2e^{2t}\sin t\\
&= t^2+1+C_1e^{2t}\cos t+C_2e^{2t}\sin t\,.
\end{aligned}$$

Using the initial condition, we find that $C_1 = -1$ and $C_2 = 3$; thus the solution to the initial value problem is given by

$$y(t) = 3e^{2t}\sin t - e^{2t}\cos t + 1 + t^2\,.$$

2. Let $L_1 = \frac{d^2}{dt^2} - 4\frac{d}{dt} + 5$ and $L_2 = \frac{d^3}{dt^3}$. Then $L_2L_1y = 0$. The zeros of the characteristic equation of $L_2L_1y = 0$ is $r = 0$ (triple roots) and $r = 2 \pm i$. Therefore, the general solution to $L_2L_1y = 0$ is

$$y(t) = C_1t^2 + C_2t + C_3 + C_4e^{2t}\cos t + C_5e^{2t}\sin t\,.$$

Then

$$\begin{aligned}
y'(t) &= 2C_1t + C_2 + (2C_4 + C_5)e^{2t}\cos t + (2C_5 - C_4)e^{2t}\sin t\,,\\
y''(t) &= 2C_1 + (3C_4 + 4C_5)e^{2t}\cos t + (3C_5 - 4C_4)e^{2t}\sin t\,;
\end{aligned}$$

thus

$$\begin{aligned}
&2C_1 + (3C_4 + 4C_5)e^{2t}\cos t + (3C_5 - 4C_4)e^{2t}\sin t\\
&\quad - 4\big(2C_1t + C_2 + (2C_4 + C_5)e^{2t}\cos t + (2C_5 - C_4)e^{2t}\sin t\big)\\
&\qquad + 5\big(C_1t^2 + C_2t + C_3 + C_4e^{2t}\cos t + C_5e^{2t}\sin t\big) = 5t^2 - 8t + 7\,.
\end{aligned}$$

Taking the initial condition into account, C_1, C_2, C_3, C_4, C_5 satisfy

$$\begin{aligned}
2C_1 - 4C_2 + 5C_3 &= 7\,,\\
-8C_1 + 5C_2 &= -8\,,\\
5C_1 &= 5\,,\\
C_3 + C_4 &= 0\,,\\
C_2 + 2C_4 + C_5 &= 1\,,
\end{aligned}$$

which implies that $(C_1, C_2, C_3, C_4, C_5) = (1, 0, 1, -1, 3)$. Therefore, the solution to the initial value problem is

$$y(t) = t^2 + 1 - e^{2t}\cos t + 3e^{2t}\sin t\,.$$

3. Assume that the solution y to the IVP above is continuously differentiable and y'' is of exponential order α for some α. Let $Y(s) = \mathscr{L}(y)(s)$. Then by the fact that the Laplace transform of $5t^2 - 8t + 7$ is $\frac{10}{s^3} - \frac{8}{s^2} + \frac{7}{s} = \frac{7s^2 - 8s + 10}{s^3}$, we find that Y satisfies

$$\begin{aligned}
Y(s) &= \frac{1}{s^2 - 4s + 5} + \frac{7s^2 - 8s + 10}{s^3(s^2 - 4s + 5)} = \frac{1}{(s-2)^2 + 1} + \frac{7s^2 - 8s + 10}{s^3(s^2 - 4s + 5)}\\
&= \mathscr{L}(e^{2t}\sin t)(s) + \frac{7s^2 - 8s + 10}{s^3(s^2 - 4s + 5)}\,.
\end{aligned}$$

Using partial fraction,

$$\frac{2}{s^3(s^2-4s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2-4s+5},$$

where A, B, C, D, E satisfy

$$\begin{aligned} A + D &= 0\,, \\ -4A + B + E &= 0\,, \\ 5A - 4B + C &= 7\,, \\ 5B - 4C &= -8\,, \\ 5C &= 10\,. \end{aligned}$$

Therefore, $(A, B, C, D, E) = (1, 0, 2, -1, 4)$ which implies that

$$\begin{aligned} Y(s) &= \mathscr{L}(e^{2t}\sin t)(s) + \mathscr{L}(1+t^2)(s) - \frac{s-2}{(s-2)^2+1} + \frac{2}{(s-2)^2+1} \\ &= \mathscr{L}(3e^{2t}\sin t)(s) + \mathscr{L}(1+t^2)(s) - \mathscr{L}(e^{2t}\cos t)(s)\,. \end{aligned}$$

By the unique inversion of Laplace transform, we find that

$$y(t) = 3e^{2t}\sin t - e^{2t}\cos t + 1 + t^2\,.$$

□

Problem 3. (15pts) Find the Laplace transform of the function $f(t) = te^t\cos t$.

Solution: Since the Fourier transform of the function $e^t\cos t$ is $\frac{s-1}{(s-1)^2+1}$, by Theorem 6.22 in the lecture note, we find that

$$\begin{aligned} \mathscr{L}(f)(s) &= -\frac{d}{ds}\frac{s-1}{(s-1)^2+1} = -\frac{(s-1)^2+1-2(s-1)(s-1)}{\big[(s-1)^2+1\big]^2} \\ &= \frac{(s-1)^2-1}{\big[(s-1)^2+1\big]^2} = \frac{s^2-2s}{\big[(s-1)^2+1\big]^2}\,. \end{aligned}$$

□

Problem 4. Complete the following.

1. (5%) Show that if f is piecewise continuous and of exponential order α for some α, then $\lim\limits_{s\to\infty}\mathscr{L}(f)(s) = 0$, where $\mathscr{L}(f)$ is the Laplace transform of f.
2. (15%) Use the Laplace transform to find the solution to the initial value problem

$$y'' + ty' - y = 0\,, \qquad y(0) = 0\,, \quad y'(0) = 3\,.$$

Solution:

1. Since f is of exponential order α, there exists M such that $\big|f(t)\big| \leqslant Me^{\alpha t}$ for all $t > 0$. Therefore,

$$\begin{aligned} \big|\mathscr{L}(f)(s)\big| &= \Big|\int_0^\infty e^{-st}f(t)\,dt\Big| \leqslant \int_0^\infty e^{-st}\big|f(t)\big|\,dt \leqslant \int_0^\infty e^{-st}Me^{\alpha t}\,dt \\ &= M\int_0^\infty e^{(\alpha-s)t}\,dt \leqslant \frac{M}{s-\alpha} \qquad \forall\, s > \alpha\,. \end{aligned}$$

As $s \to \infty$, the Sandwich lemma implies that $\lim\limits_{s\to\infty}\mathscr{L}(f)(s) = 0$.

2. Assume that y is continuously differentiable of exponential order α for some $\alpha > 0$, and y'' is piecewise continuous on $[0, \infty)$. Let $Y(s) = \mathscr{L}(y)(s)$. Then

$$s^2 Y(s) - 3 - \big[sY(s)\big]' - Y(s) = 0 \qquad \forall\, s > \alpha\,.$$

Therefore,

$$-sY'(s) + (s^2 - 2)Y(s) - 3 = 0$$

which can be further reduced to

$$Y'(s) + \big(\frac{2}{s} - s\big)Y(s) = -\frac{3}{s}\,.$$

Using the integrating factor $s^2 e^{-\frac{s^2}{2}}$, we find that

$$\big[s^2 e^{-\frac{s^2}{2}} Y(s)\big]' = -3se^{-\frac{s^2}{2}}\,;$$

thus

$$s^2 e^{-\frac{s^2}{2}} Y(s) = 3e^{-\frac{s^2}{2}} + C\,.$$

Therefore, $Y(s) = \dfrac{3}{s^2} + Ce^{\frac{s^2}{2}}$. By 1, $\lim\limits_{s\to\infty} Y(s) = 0$; thus $C = 0$. This shows that $y(t) = 3t$. $\square$