Differential Equations MA2042 Midterm Exam 1

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Problem 1. (15%) Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$, then the third order equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0 \tag{0.1}$$

corresponds to the system

$$x_1' = x_2\,, \tag{0.2a}$$
$$x_2' = x_3\,, \tag{0.2b}$$
$$x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3\,. \tag{0.2c}$$

Show that if $\{y_1, y_2, y_3\}$ and $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3\}$ are fundamental sets of equation (0.1) and (0.2), respectively, then $W[y_1, y_2, y_3](t) = c\,\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t)$, where c is a non-zero constant and W and W denote the Wronskian functions given by

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \quad \text{and} \quad \mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t) = \det\big([\boldsymbol{\varphi}_1 \vdots \boldsymbol{\varphi}_2 \vdots \boldsymbol{\varphi}_3]\big).$$

Proof. Write (0.2) as $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$, where $\boldsymbol{P}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{bmatrix}$. In the proof of Theorem 6.11 in the lecture note, we have shown that

$$\frac{d}{dt}\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t) = \mathrm{tr}(\boldsymbol{P})\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t) = -p(t)\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t)\,,$$

while Theorem 4.3 shows that

$$\frac{d}{dt}W[y_1, y_2, y_3](t) = -p(t)W[y_1, y_2, y_3](t)\,.$$

Therefore, by the fact that $W[y_1, y_2, y_3]$ and $\mathrm{W}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3]$ never vanish (due to the fact that $\{y_1, y_2, y_3\}$ and $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3\}$ are fundamental sets of corresponding ODEs), we have

$$\frac{1}{W[y_1, y_2, y_3](t)}\frac{dW[y_1, y_2, y_2](t)}{dt} = \frac{1}{\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t)}\frac{d\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t)}{dt}\,;$$

thus $\log W[y_1, y_2, y_3](t) = \log \mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t) + C$ which further implies that $W[y_1, y_2, y_3](t) = c\,\mathrm{W}[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3](t)$ for some non-zero constant c. □

Problem 2. (15%) Let $\omega \neq 0$ be a real number. Consider the initial value problem

$$y'' + \omega^2 y = 0\,, \qquad y(0) = y_0, \quad y'(0) = y_1\,.$$

Let $x_1 = y$ and $x_2 = y'$. For $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$, $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. Find the matrix $\boldsymbol{A}$ and solve the initial value problem by finding $\exp(\boldsymbol{A}t)$.

Proof. If $\boldsymbol{x} = (y, y')^{\mathrm{T}}$, then $\boldsymbol{x}' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \boldsymbol{x}$; thus $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$.

1. **Computing** $\exp(\boldsymbol{A}t)$ **by diagonalization**: The two eigenvalues of $\boldsymbol{A}$ are $\pm i\omega$ and the corresponding eigenvectors are $(\mp i, \omega)^{\mathrm{T}}$. In other words,

$$\boldsymbol{A} = \begin{bmatrix} -i & i \\ \omega & \omega \end{bmatrix} \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} \begin{bmatrix} -i & i \\ \omega & \omega \end{bmatrix}^{-1}$$

which implies that

$$\begin{aligned}
\exp(\boldsymbol{A}t) &= \begin{bmatrix} -i & i \\ \omega & \omega \end{bmatrix} \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \begin{bmatrix} -i & i \\ \omega & \omega \end{bmatrix}^{-1} = \frac{-1}{2\omega i} \begin{bmatrix} -i & i \\ \omega & \omega \end{bmatrix} \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \begin{bmatrix} \omega & -i \\ -\omega & -i \end{bmatrix} \\
&= \frac{-1}{2\omega i} \begin{bmatrix} -i & i \\ \omega & \omega \end{bmatrix} \begin{bmatrix} \omega e^{i\omega t} & -ie^{i\omega t} \\ -\omega e^{-i\omega t} & -ie^{-i\omega t} \end{bmatrix} = \frac{-1}{2\omega i} \begin{bmatrix} -i\omega(e^{i\omega t} + e^{-i\omega t}) & e^{-i\omega t} - e^{i\omega t} \\ \omega^2(e^{i\omega t} - e^{-i\omega t}) & -i\omega(e^{i\omega t} + e^{-i\omega t}) \end{bmatrix} \\
&= \frac{-1}{2\omega i} \begin{bmatrix} -2\omega i \cos\omega t & -2i\sin\omega t \\ 2i\omega^2 \sin\omega t & -2\omega i \cos\omega t \end{bmatrix} = \cos\omega t \mathbf{I} + \frac{\sin\omega t}{\omega}\boldsymbol{A}\,.
\end{aligned}$$

2. **Computing** $\exp(\boldsymbol{A}t)$ **by finding** $\boldsymbol{A}^k$: Observing that

$$\boldsymbol{A}^2 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} = \begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix} = -\omega^2 \mathbf{I}\,;$$

thus

$$\begin{aligned}
\exp(\boldsymbol{A}t) &= \mathbf{I} + \sum_{k=1}^{\infty} \frac{\boldsymbol{A}^k t^k}{k!} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\boldsymbol{A}^{2k} t^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^{2k+1} t^{2k+1}}{(2k+1)!} \\
&= \Big(1 + \sum_{k=1}^{\infty} \frac{(-\omega^2)^k t^{2k}}{(2k)!}\Big)\mathbf{I} + \sum_{k=0}^{\infty} \frac{(-\omega^2)^k t^{2k+1}}{(2k+1)!}\boldsymbol{A} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (\omega t)^{2k}}{(2k)!}\mathbf{I} + \frac{1}{\omega}\sum_{k=0}^{\infty} \frac{(-1)^k (\omega t)^{2k+1}}{(2k+1)!}\boldsymbol{A} = \cos\omega t \mathbf{I} + \frac{\sin\omega t}{\omega}\boldsymbol{A}\,.
\end{aligned}$$

Therefore, the solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0 = (y_0, y_1)^{\mathrm{T}}$ is given by

$$\boldsymbol{x}(t) = \exp(\boldsymbol{A}t)\boldsymbol{x}_0 = \Big(\cos\omega t \mathbf{I} + \frac{\sin\omega t}{\omega}\boldsymbol{A}\Big)\boldsymbol{x}_0 = \cos\omega t \boldsymbol{x}_0 + \frac{\sin\omega t}{\omega}\boldsymbol{A}\boldsymbol{x}_0 = \begin{bmatrix} y_0 \cos\omega t + y_1 \dfrac{\sin\omega t}{\omega} \\ y_1 \cos\omega t - \omega^2 y_0 \dfrac{\sin\omega t}{\omega} \end{bmatrix}.$$

Therefore, the solution to the ODE is $y(t) = y_0 \cos\omega t + y_1 \dfrac{\sin\omega t}{\omega}$. □

Problem 3. Let $\boldsymbol{A} = \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix}$.

1. (15%) Find a Jordan decomposition of $\boldsymbol{A}$.

2. (10%) Find the general solution to the ODE $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$.

Proof. 1. The character equation of $\boldsymbol{A}$ is

$$\begin{aligned} 0 &= \det(\boldsymbol{A} - \lambda \boldsymbol{I}) \\ &= \lambda^4 - (0+1-1+4)\lambda^3 + \left(\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ -2 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 0 & -3 \\ -2 & 1 \end{vmatrix} \right) \lambda^2 \\ &\quad - \left(\begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ -3 & 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 \\ -2 & -1 & 2 \\ -2 & 1 & 4 \end{vmatrix} + \begin{vmatrix} 0 & -3 & 2 \\ -2 & 1 & 2 \\ -2 & -3 & 4 \end{vmatrix} + \begin{vmatrix} 0 & -3 & 1 \\ -2 & 1 & -1 \\ -2 & 1 & -1 \end{vmatrix} \right) \lambda + \det(\boldsymbol{A}) \\ &= \lambda^4 - 4\lambda^3 + (-6+10+0+4+2-6)\lambda^2 - (0-4+4+0)\lambda + 0 = (\lambda-2)^2\lambda^2 \,. \end{aligned}$$

Therefore, the eigenvalues of $\boldsymbol{A}$ is 2 and 0, both of them are repeated double roots. Two eigenvector associated with 2 are $\boldsymbol{v}_1 = (1,0,0,1)^{\mathrm{T}}$ and $\boldsymbol{v}_2 = (0,1,1,1)^{\mathrm{T}}$, while an eigenvector associated with 0 is $(1,1,1,1)^{\mathrm{T}}$. Since

$$(\boldsymbol{A} - 0\boldsymbol{I})^2 = \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -8 & 4 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & -8 & 4 & 8 \end{bmatrix},$$

$\boldsymbol{v}_4 = (0,-1,-2,0)^{\mathrm{T}} \in \mathrm{Ker}\big((\boldsymbol{A} - 0\boldsymbol{I})^2\big) \backslash \mathrm{Ker}(\boldsymbol{A} - 0\boldsymbol{I})$. Let $\boldsymbol{v}_3 = (\boldsymbol{A} - 0\boldsymbol{I})\boldsymbol{v}_3 = (1,1,1,1)^{\mathrm{T}}$. Then a Jordan decomposition of $\boldsymbol{A}$ is

$$\boldsymbol{A} = \big[\boldsymbol{v}_1 \vdots \boldsymbol{v}_2 \vdots \boldsymbol{v}_3 \vdots \boldsymbol{v}_4\big] \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \big[\boldsymbol{v}_1 \vdots \boldsymbol{v}_2 \vdots \boldsymbol{v}_3 \vdots \boldsymbol{v}_4\big]^{-1} .$$

2. Using the Jordan decomposition obtained in 1, we have

$$\exp(\boldsymbol{A}t) = \big[\boldsymbol{v}_1 \vdots \boldsymbol{v}_2 \vdots \boldsymbol{v}_3 \vdots \boldsymbol{v}_4\big] \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \big[\boldsymbol{v}_1 \vdots \boldsymbol{v}_2 \vdots \boldsymbol{v}_3 \vdots \boldsymbol{v}_4\big]^{-1};$$

thus the general solution to $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is

$$\begin{aligned} \boldsymbol{x}(t) &= \big[\boldsymbol{v}_1 \vdots \boldsymbol{v}_2 \vdots \boldsymbol{v}_3 \vdots \boldsymbol{v}_4\big] \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \\ &= \big[\boldsymbol{v}_1 \vdots \boldsymbol{v}_2 \vdots \boldsymbol{v}_3 \vdots \boldsymbol{v}_4\big] \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{2t} \\ C_3 + C_4 t \\ C_4 \end{bmatrix} = C_1 \boldsymbol{v}_1 e^{2t} + C_2 \boldsymbol{v}_2 e^{2t} + (C_3 + C_4 t)\boldsymbol{v}_3 + C_4 \boldsymbol{v}_4 \,. \end{aligned}$$

□

Problem 4. Let $\boldsymbol{P}(t) = \frac{1}{t}\begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$.

1. (15%) Find the solution $\boldsymbol{\Phi}$ to $\boldsymbol{\Phi}' = \boldsymbol{P}(t)\boldsymbol{\Phi}$ satisfying the initial condition $\boldsymbol{\Phi}(1) = \boldsymbol{I}_2$, where $\boldsymbol{I}_2$ is the 2×2 identity matrix. (**Hint**: Consider the Euler equation $t\boldsymbol{x}' = t\boldsymbol{P}(t)\boldsymbol{x}$)

2. (15%) Find the general solution of the ODE $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{f}(t)$, where $\boldsymbol{f}(t)$ is given by

$$\boldsymbol{f}(t) = \begin{bmatrix} 4t^4 \\ 0 \end{bmatrix}.$$

Proof. 1. Let $\boldsymbol{A} = t\boldsymbol{P}(t)$. Then $\boldsymbol{A}$ is a constant matrix. The characteristic equation of $\boldsymbol{A}$ is

$$0 = \det(\boldsymbol{A} - \lambda \boldsymbol{I}_2) = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2);$$

thus the eigenvalues of $\boldsymbol{A}$ is $\lambda_1 = 4$ and $\lambda_2 = 2$. An eigenvector associated with λ_1 is $\boldsymbol{v}_1 = (3, -1)^{\mathrm{T}}$, and and eigenvector associated with λ_2 is $\boldsymbol{v}_2 = (1, -1)^{\mathrm{T}}$. Therefore, the general solution to $t\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ (which is equivalent to that $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ when $t \neq 0$) can be written as

$$\boldsymbol{x}(t) = C_1 \boldsymbol{v}_1 t^{\lambda_1} + C_2 \boldsymbol{v}_2 t^{\lambda_2} = C_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} t^4 + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} t^2.$$

A fundamental matrix $\boldsymbol{\Psi}$ of the ODE $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ is

$$\boldsymbol{\Psi}(t) = \left[\boldsymbol{v}_1 t^4 \vdots \boldsymbol{v}_2 t^2\right] = \begin{bmatrix} 3t^4 & t^2 \\ -t^4 & -t^2 \end{bmatrix};$$

thus the desired matrix $\boldsymbol{\Phi}$ is obtained by

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(1)^{-1} = \begin{bmatrix} 3t^4 & t^2 \\ -t^4 & -t^2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 3t^4 - t^2 & 3t^4 - 3t^2 \\ -t^4 + t^2 & -t^4 + 3t^2 \end{bmatrix}.$$

2. (a) **Method 1 (Variation of Parameters)**: Assume that a particular solution to $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{f}(t)$ is

$$\boldsymbol{x}_p(t) = u_1(t)\boldsymbol{v}_1 t^4 + u_2(t)\boldsymbol{v}_2 t^2.$$

Then (u_1, u_2) satisfies

$$\left[\boldsymbol{v}_1 t^4 \vdots \boldsymbol{v}_2 t^2\right] \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \boldsymbol{\Psi}(t) \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \boldsymbol{f}(t);$$

thus

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} t^{-4} & t^{-4} \\ -t^{-2} & -3t^{-2} \end{bmatrix} \begin{bmatrix} 4t^4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2t^2 \end{bmatrix}.$$

Therefore, a particular solution is

$$\boldsymbol{x}_p(t) = 2t\boldsymbol{v}_1 t^4 - \frac{2}{3}t^3 \boldsymbol{v}_2 t^2,$$

and the general solution is given by

$$\boldsymbol{x}(t) = (C_1 + 2t)\boldsymbol{v}_1 t^4 + \left(C_2 - \frac{2}{3}t^3\right)\boldsymbol{v}_2 t^2.$$

(b) **Method 2 (Using the representation formula)**: Using the representation formula for the solution to non-homogeneous equations, we find that the solution to $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{f}(t)$ with initial condition $\boldsymbol{x}(1) = \boldsymbol{x}_0$ can be written as

$$\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(1)^{-1}\boldsymbol{x}_0 + \int_1^t \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(s)^{-1}\boldsymbol{f}(s)\,ds \\
&= \big[\boldsymbol{v}_1 t^4 \vdots \boldsymbol{v}_2 t^2\big] \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \end{bmatrix} + \frac{1}{2}\big[\boldsymbol{v}_1 t^4 \vdots \boldsymbol{v}_2 t^2\big] \int_1^t \begin{bmatrix} s^{-4} & s^{-4} \\ -s^{-2} & -3s^{-2} \end{bmatrix} \begin{bmatrix} 4s^4 \\ 0 \end{bmatrix} ds \\
&= \widetilde{C}_1 \boldsymbol{v}_1 t^4 + \widetilde{C}_2 \boldsymbol{v}_2 t^2 + \big[\boldsymbol{v}_1 t^4 \vdots \boldsymbol{v}_2 t^2\big] \int_1^t \begin{bmatrix} 2 \\ -2s^2 \end{bmatrix} ds \\
&= \widetilde{C}_1 \boldsymbol{v}_1 t^4 + \widetilde{C}_2 \boldsymbol{v}_2 t^2 + \big[\boldsymbol{v}_1 t^4 \vdots \boldsymbol{v}_2 t^2\big] \begin{bmatrix} 2(t-1) \\ -\frac{2}{3}(t^3 - 1) \end{bmatrix} \\
&= (C_1 + 2t)\boldsymbol{v}_1 t^4 + \big(C_2 - \frac{2}{3}t^3\big)\boldsymbol{v}_2 t^2 \,,
\end{aligned}$$

in which $[\widetilde{C}_1, \widetilde{C}_2]^{\mathrm{T}} = \boldsymbol{\Psi}(1)^{-1}\boldsymbol{x}_0$ and $C_1 = \widetilde{C}_1 - 2$ and $C_2 = \widetilde{C}_2 + \frac{2}{3}$. □

Problem 5. To solve a first order equation $x' = f(t,x)$ with initial condition $x(t_0) = x_0$ numerically, one can use the improved Euler method which is the iteration method given by the

$$x_{n+1} = x_n + \frac{h}{2}\Big[f(t_n, x_n) + f\big(t_{n+1}, x_n + hf(t_n, x_n)\big)\Big],$$

where with h denoting the step size, $t_n = t_0 + nh$.

1. (15%) Use the improved Euler method to solve $x' = x + 1$ with $x(0) = x_0$ and show that for each fixed $t = nh$ (which implies that $n \to \infty$ as the step size $h \to 0$), one has $x_n \to (x_0+1)e^t - 1$ as $h \to 0$.

2. (10%) Compute the local truncation error $\tau_n(h)$ and show that

$$|\tau_n(h)| \leqslant \frac{e^T|x_0+1|}{6}h^2 \qquad \forall\, n \in \Big\{0, 1, \cdots, \frac{T}{h} - 1\Big\}. \tag{0.3}$$

(**Note**: You cannot apply the theorem taught in class since the corresponding Φ here is not bounded on $\mathbb{R}$. Write down the numerical scheme and see if the sequence $\{x_n\}_{n=1}^N$ produced by the scheme converges.)

Proof. 1. Let $T > 0$ be given, and $N = T/h$. Since $f(y) = y + 1$, using the improved Euler we have

$$\begin{aligned} x_{n+1} &= x_n + \frac{h}{2}\big[(x_n+1) + x_n + h(x_n+1) + 1\big] = x_n + \frac{h}{2}(2+h)(x_n+1) \\ &= \Big(1 + h + \frac{h^2}{2}\Big)x_n + \frac{h(2+h)}{2}. \end{aligned}$$

As a consequence,

$$\begin{aligned} x_n &= \Big(1+h+\frac{h^2}{2}\Big)x_{n-1} + \frac{h(2+h)}{2}, \\ \Big(1+h+\frac{h^2}{2}\Big)x_{n-1} &= \Big(1+h+\frac{h^2}{2}\Big)^2 x_{n-2} + \frac{h(2+h)}{2}\Big(1+h+\frac{h^2}{2}\Big), \\ \Big(1+h+\frac{h^2}{2}\Big)^2 x_{n-2} &= \Big(1+h+\frac{h^2}{2}\Big)^3 x_{n-3} + \frac{h(2+h)}{2}\Big(1+h+\frac{h^2}{2}\Big)^2, \\ \vdots \quad &= \quad \vdots \\ \Big(1+h+\frac{h^2}{2}\Big)^{n-1} x_1 &= \Big(1+h+\frac{h^2}{2}\Big)^n x_0 + \frac{h(2+h)}{2}\Big(1+h+\frac{h^2}{2}\Big)^n. \end{aligned}$$

Summing all the equalities above, we find that

$$\begin{aligned} x_n &= \Big(1+h+\frac{h^2}{2}\Big)^n x_0 + \frac{h(2+h)}{2}\sum_{k=0}^{n}\Big(1+h+\frac{h^2}{2}\Big)^k \\ &= \Big(1+h+\frac{h^2}{2}\Big)^n x_0 + \frac{h(2+h)}{2}\,\frac{\Big(1+h+\frac{h^2}{2}\Big)^{n+1} - 1}{h + \frac{h^2}{2}} \\ &= \Big(1+h+\frac{h^2}{2}\Big)^n x_0 + \Big(1+h+\frac{h^2}{2}\Big)^{n+1} - 1. \end{aligned} \tag{0.4}$$

Since

$$\lim_{h\to 0}\Big(1+h+\frac{h^2}{2}\Big)^n = \lim_{h\to 0}\Big(1+h+\frac{h^2}{2}\Big)^{\frac{T}{h}} = \lim_{h\to 0}\Big(1+h+\frac{h^2}{2}\Big)^{\frac{T}{h+h^2/2}(1+h/2)} = e^T\,,$$

we conclude that $\lim\limits_{h\to 0} x_n = e^t x_0 + e^t - 1 = (x_0+1)e^t - 1$.

2. From the previous problem, we know that the exact solution to the ODE $x' = x+1$ with initial data $x(0) = x_0$ is $x(t) = (x_0+1)e^t - 1$. We note that the improved Euler method can be written as

$$x_{n+1} = x_n + h\Phi(h, t_n, x_n)\,,$$

where $\Phi(h,t,x) = \dfrac{(2+h)(x+1)}{2}$.

By the definition of the local truncation error,

$$\begin{aligned}\tau_n(h) &= \frac{x((n+1)h) - x(nh) - h\Phi\big(h, nh, x(nh)\big)}{h} \\ &= (x_0+1)\frac{e^{(n+1)h} - e^{nh}}{h} - \frac{2+h}{2}(x_0+1)e^{nh} \\ &= (x_0+1)e^{nh}\Big[\frac{e^h-1}{h} - 1 - \frac{h}{2}\Big]\,.\end{aligned}$$

The Taylor theorem implies that

$$\frac{e^h-1}{h} - 1 - \frac{h}{2} = \frac{h^2}{6}e^\xi$$

for some $\xi \in (0,h)$; thus $\left|\dfrac{e^h-1}{h} - 1 - \dfrac{h}{2}\right| \leqslant \dfrac{h^2}{6}e^h$ which further implies that (0.3). □