Differential Equations MA2041-A Final Exam

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1	2(1)	2(2)	2(3)	3(1)	3(2)	4	5	6(1)	6(2)	Total

Formulas:

1. The Cauchy product of two series: inside the interval of convergence,ds

$$\Big(\sum_{k=0}^{\infty} a_k x^k\Big)\Big(\sum_{k=0}^{\infty} b_k x^k\Big) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} a_{k-j} b_j\Big) x^k \,.$$

2. The following formula concerns with solving the following ODE

$$x^2 y'' + x p(x) y' + q(x) y = 0 \qquad x > 0 \,, \tag{0.1}$$

where $p(x) = \sum\limits_{k=0}^{\infty} p_k x^k$ and $q(x) = \sum\limits_{k=0}^{\infty} q_k x^k$ are two power series with non-zero radius of convergence, if $y = \varphi(r, x) = \sum\limits_{k=0}^{\infty} a_k(r) x^{k+r}$ is a solution, then

$$F(k+r) a_k(r) + \sum_{j=0}^{k-1} \big((j+r) p_{k-j} + q_{k-j}\big) a_j(r) = 0 \qquad \forall\, k \in \mathbb{N}\,, \tag{0.2}$$

where $F(r) = r(r-1) + p_0 r + q_0$ and a_0 is assumed to be a given constant. Let r_1, r_2 be two roots of $F(r) = 0$, and $r_1 > r_2$ if $r_1, r_2 \in \mathbb{R}$.

(a) If $r_1 - r_2 \notin \mathbb{N} \cup \{0\}$, then

$$y_1(x) = \sum_{k=0}^{\infty} a_k(r_1) x^{k+r_1} \quad \text{and} \quad y_2(x) = \sum_{k=0}^{\infty} a_k(r_2) x^{k+r_2}$$

are solutions to (0.1), where $\big\{a_k(r_1)\big\}_{k=1}^{\infty}$ and $\big\{a_k(r_2)\big\}_{k=1}^{\infty}$ are given by the recurrence relation (0.2).

(b) If $r_1 = r_2$, then

$$y_1(x) = \sum_{k=0}^{\infty} a_k(r_1) x^{k+r_1} \quad \text{and} \quad y_2(x) \text{ given in Problem 4}$$

are solutions to (0.1).

(c) If $r_1 - r_2 = N \in \mathbb{N}$, then two solutions of (0.1) are given by

$$y_1(x) = \sum_{k=0}^{\infty} a_k(r_1) x^{k+r_1} \quad \text{and} \quad y_2(x) = \frac{b_0}{a_0} y_1(x) \log(x) + \sum_{k=0}^{\infty} c_k(r_2) x^{k+r_2} \,,$$

where $b_0 = \lim\limits_{r \to r_2} a_N(r)$ and $c_k(r_2) = \dfrac{\partial}{\partial r}\Big|_{r=r_2} (r - r_2) a_k(r)$.

Problem 1. (20%) Assume that a series solution to $y'' - 2xy' + 10y = 0$ satisfying the initial conditions $y(0) = 1$ and $y'(0) = 0$ is $y = \sum\limits_{\ell=0}^{\infty} a_\ell x^\ell$. Show that $a_{2\ell-1} = 0$ for all $\ell \in \mathbb{N}$. Moreover, $a_{2\ell}$ is of the form

$$a_{2\ell} = c\frac{(2\ell-i)!}{(\ell-j)!(2\ell-k)!} \qquad \forall\, \ell \in \mathbb{N}\,, \ell \geqslant 4$$

for some constant c and integers i, j, ℓ. Find i, j, k as well as c.

Solution: Let $y = \sum\limits_{\ell=0}^{\infty} a_\ell x^\ell$ be the solution to the ODE above. Then

$$\begin{aligned} y' &= \sum_{\ell=0}^{\infty} \ell a_\ell x^{\ell-1}\,, \\ y'' &= \sum_{\ell=0}^{\infty} \ell(\ell-1) a_\ell x^{\ell-2} = \sum_{\ell=0}^{\infty} (\ell+2)(\ell+1) a_{\ell+2} x^\ell\,; \end{aligned}$$

thus we have

$$\sum_{\ell=0}^{\infty} \big[(\ell+2)(\ell+1)a_{\ell+2} + 2(5-\ell)a_\ell\big] x^\ell = 0\,.$$

Therefore,

$$a_{\ell+2} = \frac{2(\ell-5)}{(\ell+2)(\ell+1)} a_\ell \qquad \forall\, \ell \in \mathbb{N} \cup \{0\}\,.$$

Using the initial condition, we find that $a_0 = 1$ and $a_1 = 0$; thus the recurrence relation above implies that $a_{2\ell-1} = 0$ for all $\ell \in \mathbb{N}$. Moreover,

$$\begin{aligned} a_{2\ell} &= \frac{2(2\ell-2-5)}{(2\ell)(2\ell-1)} a_{2\ell-2} = \frac{2^2(2\ell-2-5)(2\ell-4-5)}{(2\ell)(2\ell-1)(2\ell-2)(2\ell-3)} a_{2\ell-4} = \cdots \\ &= \frac{2^\ell(2\ell-7)(2\ell-9)\cdot 1 \cdots (-1)\cdot(-3)\cdot(-5)}{(2\ell)!} a_0 \\ &= \frac{-15 \cdot 2^\ell (2\ell-7)!}{(2\ell-8)(2\ell-10)\cdots 2 \cdot (2\ell)!} = \frac{-15\cdot 2^\ell(2\ell-7)(2\ell-8)\cdots 1}{2^{\ell-4}(\ell-4)!(2\ell)!} \\ &= -240\frac{(2\ell-7)!}{(\ell-4)!(2\ell)!}\,. \end{aligned}$$

Therefore, $c = -240$ and $(i, j, k) = (7, 4, 0)$.

Problem 2. Consider the Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ for some $n \in \mathbb{N}$.

1. (5%) Find the recurrence relation of the coefficient $\{a_k\}_{k=0}^{\infty}$ of a series solution $\sum\limits_{k=0}^{\infty} a_k x^k$ about 0 has to satisfy.
2. (10%) Show that for each $n \in \mathbb{N}$, there is always a polynomial solution $y = p_n(x)$ to the Legendre equation above (using the recurrence relation obtained in Step 1).
3. (10%) Find the polynomial solution $p_5(x)$ of Legendre equation satisfying $p_5(1) = 1$.

Solution:

1. If $y = \sum\limits_{k=0}^{\infty} a_k x^k$ be a solution, then

$$\sum_{k=0}^{\infty} \Big[(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + n(n+1)a_k\Big]x^k = 0\,.$$

   Therefore, we obtain the following recurrence relation

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,. \tag{0.3}$$

2. By the recurrence relation above, we find that $a_{n+2} = 0$ and this further implies that $a_{n+2\ell} = 0$ for all $\ell \in \mathbb{N}$. Therefore,

   (a) if n is an even number, a polynomial solution is given by

$$p_n(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_n x^n\,;$$

   (b) if n is an odd number, a polynomial solution is given by

$$p_n(x) = a_1 x + a_3 x^3 + \cdots + a_n x^n\,;$$

   in which $\{a_k\}_{k=0}^{\infty}$ satisfies the recurrence relation (0.3).

3. By the analysis above, we find that

$$p_5(x) = a_1 x + a_3 x^3 + a_5 x^5\,,$$

   where $a_3 = \frac{2-30}{6} a_1 = -\frac{14}{3} a_1$ and $a_5 = \frac{12-30}{20} a_3 = \frac{21}{5} a_1$. To satisfy $p_5(1) = 1$, a_1 must satisfy

$$a_1 - \frac{14}{3} a_1 + \frac{21}{5} a_1 = 1\,;$$

   thus $a_1 = \frac{15}{8}$. Therefore,

$$p_5(x) = \frac{15}{8} x - \frac{35}{4} x^3 + \frac{63}{8} x^5\,.$$

Problem 3. Solve the differential equation

$$\frac{\sin^2(2t)}{4}y''(t) - (5\sin^3 t\cos t + 3\sin t\cos^3 t)y'(t) + 5y(t) = 0\,, \qquad 0 < t < \frac{\pi}{2} \tag{0.4}$$

following the steps below:

(1) (10%) Let $x = \tan t$ and $z(x) = y(\arctan x)$. Find the corresponding differential equation that z satisfies (the function arctan is identical to $\tan^{-1}$).

(2) (10%) Find the general solution to the equation for z, and then use it to find a solution to (0.4).

Solution:

(1) Let $x = \tan t$ and $z(x) = y(\tan^{-1} x)$. Then

$$z'(x) = y'(\tan^{-1} x)\frac{1}{1+x^2} \quad \text{and} \quad z''(x) = y''(\tan^{-1} x)\frac{1}{(1+x^2)^2} + y'(\tan^{-1} x)\frac{-2x}{(1+x^2)^2}\,.$$

Therefore,

$$y'(\tan^{-1} x) = (1+x^2)z'(x) \quad \text{and} \quad y''(\tan^{-1} x) = (1+x^2)^2 z''(x) + 2x(1+x^2)z'(x)\,.$$

Letting $t = \tan^{-1} x$ as well as $\sin t = \dfrac{x}{\sqrt{1+x^2}}$ and $\cos t = \dfrac{1}{\sqrt{1+x^2}}$ in the ODE we find that

$$y''(\tan^{-1} x)\frac{x^2}{(1+x^2)^2} - y'(\tan^{-1} x)\frac{5x^3+3x}{(1+x^2)^2} + 5y(\tan^{-1} t) = 0$$

thus

$$x^2 z''(x) - 3xz'(x) + 5z(x) = 0\,.$$

(2) Let r satisfy $r(r-1) - 3r + 5 = 0$. Then $r^2 - 4r + 5 = 0$ which implies $r = 2+i$ and $r = 2-i$. Therefore, the general solution of (0.4) is

$$z(x) = C_1 x^2 \log\cos x + C_2 x^2 \log\sin x\,.$$

Therefore,

$$y(t) = z(\tan t) = C_1 \tan^2 t\log\cos(\tan t) + C_2\tan^2 t\log\sin(\tan t)\,.$$

Problem 4. (20%) Consider solving the ODE

$$x^2y'' + xp(x)y' + q(x)y = 0 \qquad x > 0\,, \tag{0.1}$$

where $p(x) = \sum\limits_{k=0}^{\infty} p_k x^k$ and $q(x) = \sum\limits_{k=0}^{\infty} q_k x^k$ are two power series with non-zero radius of convergence. Show that if the indicial equation $r(r-1) + rp_0 + q_0 = 0$ has a double root r, then

$$y_2(x) = \log x \sum_{k=0}^{\infty} a_k(r) x^{x+r} + \sum_{k=0}^{\infty} a_k'(r) x^{k+r}$$

is a solution to (0.1) as long as the series converges in an interval, where $\{a_k(r)\}_{k=1}^{\infty}$ is a sequence satisfying the recurrence relation (0.2).

Proof. Let $y_1(x) = \sum\limits_{k=0}^{\infty} a_k(r) x^{x+r}$. Then

$$\begin{aligned}
x^2y_1'' + xp(x)y_1' + q(x)y &= \sum_{k=0}^{\infty} (k+r)(k+r-1)a_k(r)x^{k+r} + \Big(\sum_{k=0}^{\infty} p_k x^k\Big)\Big(\sum_{k=0}^{\infty} (k+r)a_k(r)x^{k+r}\Big) \\
&\quad + \Big(\sum_{k=0}^{\infty} q_k x^k\Big)\Big(\sum_{k=0}^{\infty} a_k(r)x^{k+r}\Big) \\
&= \sum_{k=0}^{\infty} \Big[(k+r)(k+r-1)a_k(r) + \sum_{j=0}^{k} p_{k-j}(j+r)a_j(r) + \sum_{j=0}^{k} q_{k-j}a_j(r)\Big] x^{k+r} \\
&= F(r)a_0 + \sum_{k=1}^{\infty} \Big[F(k+r)a_k(r) + \sum_{j=0}^{k-1} \big[(j+r)p_{k-j} + q_{k-j}\big]a_j(r)\Big] x^{k+r}\,.
\end{aligned}$$

Since $F(r) = 0$, using (0.2) we find that y_1 is also a solution to (0.1).

Differentiating (0.2) w.r.t. r variable, we find that

$$\big[2(k+r_1) - 1\big]a_k(r_1) + \sum_{j=0}^{k} p_{k-j}a_j(r_1) + \sum_{j=0}^{k} \big[p_{k-j}(j+r_1) + q_{k-j}\big]a_j'(r_1) = 0 \qquad \forall\, k \in \mathbb{N} \cup \{0\}\,.$$

As a consequence,

$$\begin{aligned}
&x^2y_2'' + xp(x)y_2' + q(x)y_2 \\
&\quad = x^2y_1''(x)\log x + 2xy_1'(x) - y_1(x) + \sum_{k=0}^{\infty} (k+r_1)(k+r-1)a_k'(r_1)x^{k+r_1} \\
&\qquad + xp(x)y_1'(x)\log x + p(x)y_1(x) + \Big(\sum_{k=0}^{\infty} p_k x^k\Big)\Big(\sum_{k=0}^{\infty} (k+r_1)a_k'(r_1)x^{k+r_1}\Big) \\
&\qquad + q(x)y_1(x)\log x + \Big(\sum_{k=0}^{\infty} q_k x^k\Big)\Big(\sum_{k=0}^{\infty} a_k'(r_1)x^{k+r_1}\Big) \\
&\quad = \sum_{k=0}^{\infty} \big[2(k+r_1) - 1\big]a_k(r_1)x^{k+r_1} + \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} p_{k-j}a_j(r_1)\Big)x^{k+r_1} \\
&\qquad + \sum_{k=0}^{\infty} \Big((k+r_1)(k+r-1)a_k'(r_1) + \sum_{j=0}^{k} \big[p_{k-j}(j+r_1) + q_{k-j}\big]a_j'(r_1)\Big)x^{k+r_1} = 0\,;
\end{aligned}$$

$y_2(x)$ is a solution to (0.1). □

Problem 5. (20%) Given a solution $J_0(x) = 1 + \sum\limits_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$ to Bessel's equation of order zero

$$x^2 y'' + xy' + x^2 y = 0\,,$$

use the method of reduction of order to show that another solution can be given by

$$y_2(x) = J_0(x) \int \frac{dx}{x|J_0(x)|^2}\,.$$

Proof. Suppose that another solution to Bessel's equation of order zero is $y_2(x) = J_0(x)v(x)$. Then

$$x^2\big(J_0(x)v(x)\big)'' + x\big(J_0(x)v(x)\big)' + x^2 J_0(x)v(x) = 0$$

which can be further reduced to

$$xJ_0(x)v''(x) + \big[2xJ_0'(x) + J_0(x)\big]v'(x) = 0$$

or

$$v''(x) + \Big[\frac{2J_0'(x)}{J_0(x)} + \frac{1}{x}\Big]v' = 0\,.$$

Therefore, the method of integrating factor shows that

$$\Big(e^{2\log J_0(x) + \log x} v'(x)\Big)' = 0$$

which further implies that

$$v'(x) = \frac{C_1}{x|J_0(x)|^2}\,.$$

As a consequence,

$$v(x) = C_1 \int \frac{dx}{x|J_0(x)|^2} + C_2$$

which implies that another independent solution can be given by $y_2(x) = J_0(x) \int \frac{dx}{x|J_0(x)|^2}$. □

Problem 6. For $\nu \geqslant 0$, the Bessel function of the first kind of order ν, denoted by J_ν, is defined as the series solution to the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

of the form $J_\nu(x) = x^\nu\Big[\frac{1}{\Gamma(\nu+1)2^\nu} + \sum_{k=1}^{\infty} a_k(\nu)x^k\Big]$, where $\Gamma : (0,\infty) \to \mathbb{R}$ is the Gamma-function which has the property that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$.

1. (15%) Show that $J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}$.

2. (10%) Verify that $J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x)$.

Proof. 1. Note that if $\sum_{k=0}^{\infty} a_k(\nu)x^{k+\nu}$ is a solution to the Bessel equation of order ν, then

$$a_k(\nu) = \frac{-1}{(k+\nu-\nu))(k+\nu+\nu)}a_{k-2}(\nu) = \frac{-1}{k(k+2\nu)}a_{k-2}(\nu)$$

and $a_1 = 0$. Therefore, $a_{2m+1} = 0$ for all $m \in \mathbb{N} \cup \{0\}$ and

$$\begin{aligned} a_{2k}(\nu) &= \frac{1}{2k(2k+2\nu)(2k-2)(2k-2+2\nu)}a_{2k-4}(\nu) = \cdots \\ &= \frac{(-1)^k}{2k(2k-2)(2k-4)\cdots 2(2k+2\nu)(2k+2\nu-2)\cdots(2+2\nu)}a_0 \\ &= \frac{(-1)^k}{2^{2k}k!(k+\nu)(k+\nu-1)\cdots(\nu+1)} \cdot \frac{1}{\Gamma(\nu+1)2^\nu}. \end{aligned}$$

Using the property that $\Gamma(x+1) = x\Gamma(x)$, we find that

$$(k+\nu)(k+\nu-1)\cdots(\nu+1)\Gamma(\nu+1) = \Gamma(k+\nu+1);$$

thus

$$a_{2k}(\nu) = \frac{(-1)^k\Gamma(\nu+1)}{2^{2k}k!\Gamma(k+\nu+1)} \cdot \frac{1}{\Gamma(\nu+1)2^\nu} = \frac{(-1)^k}{2^{2k+\nu}k!\Gamma(k+\nu+1)}.$$

Therefore,

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\nu}k!\Gamma(k+\nu+1)}x^{2k+\nu} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu}.$$

2. Using the expression of J_ν, we have

$$\begin{aligned} \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k\nu}{k!\Gamma(k+\nu+1)}\Big(\frac{x}{2}\Big)^{2k+\nu-1} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu)}\Big(\frac{x}{2}\Big)^{2k+\nu-1} \\ &= \sum_{k=0}^{\infty}\Big[\frac{(-1)^k\nu}{k!\Gamma(k+\nu+1)} - \frac{(-1)^k(k+\nu)}{k!\Gamma(k+\nu+1)}\Big]\Big(\frac{x}{2}\Big)^{2k+\nu-1} \\ &= \sum_{k=0}^{\infty}\Big[\frac{(-1)^{k+1}k}{k!\Gamma(k+\nu+1)}\Big]\Big(\frac{x}{2}\Big)^{2k+\nu-1} = \sum_{k=1}^{\infty}\Big[\frac{(-1)^{k+1}k}{k!\Gamma(k+\nu+1)}\Big]\Big(\frac{x}{2}\Big)^{2k+\nu-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+(\nu+1)+1)}\Big(\frac{x}{2}\Big)^{2k+(\nu+1)} = J_{\nu+1}(x). \end{aligned}$$

□