Mathematical Modeling MA3067-* Midterm

National Central University, Dec. 14, 2022

Problem 1. (16pts) 在一支點懸掛一個彈簧，彈簧末端系一質量為 m 的物體，這就構成了一個彈簧擺（如 Figure 1 所示）。假設彈簧無質量、虎克常數是 k 且在不受力的情況下長度是 L，而物體在垂直於地面（平行於重力方向）的平面上運動。若在時間 t 時物體到支點的距離為 $r(t)$ 而垂線到物體與支點連線的方向角為 $\theta(t)$（逆時針方向為正）。證明 r 與 θ 滿足

$$\begin{aligned} r'' - r(\theta')^2 &= g\cos\theta - \frac{k}{m}(r-L)\,, \\ 2r'\theta' + r\theta'' &= -g\sin\theta\,. \end{aligned}$$

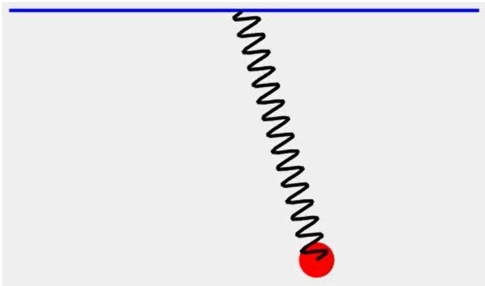

Figure 1: 彈簧擺

Problem 2. Solve the initial value problem

$$x'\cos t + x\sin t = t\,, \qquad x(0) = x_0 \tag{$\star$}$$

by the methods mentioned below.

1. (16pts) Solve ($\star$) using the method of integrating factor. Do **NOT** apply any formula for the solution to IVP but instead find the integrating factor and then the solution step by step.
2. (16pts) Solve ($\star$) by differentiating ($\star$) and solve the resulting second order initial value problem by the method of variation of parameters.

Problem 3. (16pts) Given one solution $\varphi_1(t) = 1 + t^2$ of the ODE

$$(1+t^2)x''(t) - 2x(t) = 0\,, \tag{$\star\star$}$$

find another solution φ_2 to ($\star\star$) which is linearly independent of φ_1. Do **NOT** use the formula

$$\varphi_2(t) = \varphi_1(t)\int \frac{1}{\varphi_1(t)^2} e^{-B(t)}\,dt\,.$$

Problem 4. (20pts) Find the solution to the initial value problem

$$x'' - 3x' - 4x = 30e^{2t}\,, \qquad x(0) = x'(0) = 0 \tag{$\star\star\star$}$$

by rewriting ($\star\star\star$) into a linear system $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}(t)$ with initial condition $\boldsymbol{y}(0) = \boldsymbol{y}_0$ and making use of $e^{t\boldsymbol{A}}$.

Problem 5. (16pts) Find the solution to the initial value problem

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{f}(t)\,, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0\,,$$

where $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$, $\boldsymbol{f}(t) = \begin{bmatrix} e^t \\ 0 \\ -t^2 \\ -2t \\ -2 \end{bmatrix}$ and $\boldsymbol{x}_0 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.