數學建模 MA3067-*

Chapter A. Vector Calculus（向量微積分）

§A.1 Vector Fields

Definition (Vector Fields - 向量場)

A (two-dimensional) vector field over a plane region R is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y) \in \mathbb{R}^2$ to each point (x, y) in R. A (three-dimensional) vector field over a solid region Q is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y, z) \in \mathbb{R}^3$ to each point (x, y, z) in Q.

In general, an n-dimensional vector field over a region $D \subseteq \mathbb{R}^n$ is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ to each point $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ in D.

§A.1 Vector Fields

Definition (Vector Fields - 向量場)

A (two-dimensional) vector field over a plane region R is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y) \in \mathbb{R}^2$ to each point (x, y) in R. A (three-dimensional) vector field over a solid region Q is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y, z) \in \mathbb{R}^3$ to each point (x, y, z) in Q.

In general, an n-dimensional vector field over a region $D \subseteq \mathbb{R}^n$ is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ to each point $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ in D.

§A.1 Vector Fields

Definition (Vector Fields - 向量場)

A (two-dimensional) vector field over a plane region R is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y) \in \mathbb{R}^2$ to each point (x, y) in R. A (three-dimensional) vector field over a solid region Q is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x, y, z) \in \mathbb{R}^3$ to each point (x, y, z) in Q.

In general, an n-dimensional vector field over a region $D \subseteq \mathbb{R}^n$ is a vector-valued function $\boldsymbol{F}$ that assigns a vector $\boldsymbol{F}(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ to each point $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ in D.

§A.1 Vector Fields

Definition (Vorticity - 旋度)

Let Q be an open region in space, and $\boldsymbol{F}: Q \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$. The curl of $\boldsymbol{F}$, also called the vorticity of $\boldsymbol{F}$, is a vector field given by

$$\mathrm{curl}\boldsymbol{F} = \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} - \Big(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k}\,.$$

If $\mathrm{curl}\boldsymbol{F} = \mathbf{0}$, then $\boldsymbol{F}$ is said to be ***irrotational***.

Symbolically, the curl of $\boldsymbol{F}$ is given by

$$\mathrm{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}\,.$$

§A.1 Vector Fields

Definition (Vorticity - 旋度)

Let Q be an open region in space, and $\boldsymbol{F}: Q \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$. The curl of $\boldsymbol{F}$, also called the vorticity of $\boldsymbol{F}$, is a vector field given by

$$\mathrm{curl}\boldsymbol{F} = \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} - \Big(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k}\,.$$

If $\mathrm{curl}\boldsymbol{F} = \mathbf{0}$, then $\boldsymbol{F}$ is said to be ***irrotational***.

Symbolically, the curl of $\boldsymbol{F}$ is given by

$$\mathrm{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}.$$

§A.1 Vector Fields

Remark: Let $\boldsymbol{F}$ be a two dimensional vector field given by $\boldsymbol{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. We can also define the curl of $\boldsymbol{F}$ by treating $\boldsymbol{F}$ as a three-dimensional vector field

$$\widetilde{\boldsymbol{F}}(x, y, z) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} + 0\mathbf{k}$$

(which is a three-dimensional vector field independent of z) and define $\mathrm{curl}\boldsymbol{F}$ as the third component of $\mathrm{curl}\widetilde{\boldsymbol{F}}$ (for the first two components of $\mathrm{curl}\widetilde{\boldsymbol{F}}$ are zero). Therefore, the curl of a two dimensional vector field $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$ is a scalar function given by

$$\mathrm{curl}\boldsymbol{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Moreover, by defining the differential operator $\nabla^{\perp} = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$ on plane we have the symbolic representation

$$\mathrm{curl}\boldsymbol{F} = \nabla^{\perp} \cdot \boldsymbol{F}.$$

§A.1 Vector Fields

Remark: Let $\boldsymbol{F}$ be a two dimensional vector field given by $\boldsymbol{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. We can also define the curl of $\boldsymbol{F}$ by treating $\boldsymbol{F}$ as a three-dimensional vector field

$$\widetilde{\boldsymbol{F}}(x,y,z) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} + 0\mathbf{k}$$

(which is a three-dimensional vector field independent of z) and define $\mathrm{curl}\boldsymbol{F}$ as the third component of $\mathrm{curl}\widetilde{\boldsymbol{F}}$ (for the first two components of $\mathrm{curl}\widetilde{\boldsymbol{F}}$ are zero). Therefore, the curl of a two dimensional vector field $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$ is a scalar function given by

$$\mathrm{curl}\boldsymbol{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Moreover, by defining the differential operator $\nabla^{\perp} = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$ on plane we have the symbolic representation

$$\mathrm{curl}\boldsymbol{F} = \nabla^{\perp} \cdot \boldsymbol{F}.$$

§A.1 Vector Fields

Remark: Let $\boldsymbol{F}$ be a two dimensional vector field given by $\boldsymbol{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$. We can also define the curl of $\boldsymbol{F}$ by treating $\boldsymbol{F}$ as a three-dimensional vector field

$$\widetilde{\boldsymbol{F}}(x, y, z) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j} + 0\mathbf{k}$$

(which is a three-dimensional vector field independent of z) and define $\mathrm{curl}\boldsymbol{F}$ as the third component of $\mathrm{curl}\widetilde{\boldsymbol{F}}$ (for the first two components of $\mathrm{curl}\widetilde{\boldsymbol{F}}$ are zero). Therefore, the curl of a two dimensional vector field $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$ is a scalar function given by

$$\mathrm{curl}\boldsymbol{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,.$$

Moreover, by defining the differential operator $\nabla^{\perp} = \left(-\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial x}\right)$ on plane we have the symbolic representation

$$\mathrm{curl}\boldsymbol{F} = \nabla^{\perp} \cdot \boldsymbol{F}\,.$$

§A.1 Vector Fields

Definition (Divergence - 散度)

Let R be an open region in the plane, and $\boldsymbol{F}: R \to \mathbb{R}^2$ be a vector field given by $\boldsymbol{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\mathrm{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Let Q be an open region in space, and $\boldsymbol{F}: Q \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\mathrm{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The symbolic representation of the divergence is

$$\mathrm{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F}.$$

§A.1 Vector Fields

Definition (Divergence - 散度)

Let R be an open region in the plane, and $\boldsymbol{F} : R \to \mathbb{R}^2$ be a vector field given by $\boldsymbol{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\mathrm{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Let Q be an open region in space, and $\boldsymbol{F} : Q \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\mathrm{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

The symbolic representation of the divergence is

$$\mathrm{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F}.$$

§A.1 Vector Fields

Definition (Divergence - 散度)

Let R be an open region in the plane, and $\boldsymbol{F} : R \to \mathbb{R}^2$ be a vector field given by $\boldsymbol{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\operatorname{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} .$$

Let Q be an open region in space, and $\boldsymbol{F} : Q \to \mathbb{R}^3$ be a vector field given by $\boldsymbol{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. The divergence of $\boldsymbol{F}$ is a scalar function given by

$$\operatorname{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} .$$

The symbolic representation of the divergence is

$$\operatorname{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F} .$$

§A.1 Vector Fields

Remark: In general, if D is an open region in $\mathbb{R}^n$ and $\boldsymbol{F} : D \to \mathbb{R}^n$ be a vector field given by $\boldsymbol{F}(\boldsymbol{x}) = \big(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots, F_n(\boldsymbol{x})\big)$, where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, the divergence of $\boldsymbol{F}$ is a scalar function given by

$$\mathrm{div}\boldsymbol{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Theorem

Let $\boldsymbol{F}$ be a three-dimensional vector field given by $\boldsymbol{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. If M, N, P have continuous second partial derivatives, then

$$\mathrm{div}(\mathrm{curl}\boldsymbol{F}) = 0.$$

§A.1 Vector Fields

Remark: In general, if D is an open region in $\mathbb{R}^n$ and $\boldsymbol{F}: D \to \mathbb{R}^n$ be a vector field given by $\boldsymbol{F}(\boldsymbol{x}) = \big(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \cdots, F_n(\boldsymbol{x})\big)$, where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, the divergence of $\boldsymbol{F}$ is a scalar function given by

$$\mathrm{div}\boldsymbol{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

Theorem

Let $\boldsymbol{F}$ be a three-dimensional vector field given by $\boldsymbol{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. If M, N, P have continuous second partial derivatives, then

$$\mathrm{div}(\mathrm{curl}\boldsymbol{F}) = 0.$$

§A.2 Line Integrals

§A.2.1 Curves and parametric equations

Definition

A subset C in the plane (or space) is called a ***curve*** if C is the image of an interval $I \subseteq \mathbb{R}$ under a continuous vector-valued function $\boldsymbol{r}$. The continuous function $\boldsymbol{r} : I \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a ***parametrization*** of the curve, and the equation

$$(x, y) = \boldsymbol{r}(t)\,,\ \ t \in I \qquad \big(\text{or } (x, y, z) = \boldsymbol{r}(t),\ \ t \in I\big)$$

is called a ***parametric equation*** of the curve.

A curve C is called a ***plane curve*** if it is a subset in the plane.

§A.2 Line Integrals

Definition

A curve C is called ***simple*** if it has an injective parametrization; that is, there exists $\boldsymbol{r}: I \to \mathbb{R}^3$ such that $\boldsymbol{r}(I) = C$ and $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ implies that $x = y$. A curve C with parametrization $\boldsymbol{r}: I \to \mathbb{R}^3$ is called ***closed*** if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. A ***simple closed*** curve C is a closed curve with parametrization $\boldsymbol{r}: [a, b] \to \mathbb{R}^3$ such that $\boldsymbol{r}$ is one-to-one on $[a, b)$. A ***smooth*** curve C is a curve with continuously differentiable parametrization $\boldsymbol{r}: I \to \mathbb{R}^3$ such that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

§A.2 Line Integrals

Definition

A curve C is called ***simple*** if it has an injective parametrization; that is, there exists $\boldsymbol{r}: I \to \mathbb{R}^3$ such that $\boldsymbol{r}(I) = C$ and $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ implies that $x = y$. A curve C with parametrization $\boldsymbol{r}: I \to \mathbb{R}^3$ is called ***closed*** if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. A ***simple closed*** curve C is a closed curve with parametrization $\boldsymbol{r}: [a, b] \to \mathbb{R}^3$ such that $\boldsymbol{r}$ is one-to-one on $[a, b)$. A ***smooth*** curve C is a curve with continuously differentiable parametrization $\boldsymbol{r}: I \to \mathbb{R}^3$ such that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

§A.2 Line Integrals

Definition

A curve C is called ***simple*** if it has an injective parametrization; that is, there exists $\boldsymbol{r} : I \to \mathbb{R}^3$ such that $\boldsymbol{r}(I) = C$ and $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ implies that $x = y$. A curve C with parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ is called ***closed*** if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. A ***simple closed*** curve C is a closed curve with parametrization $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ such that $\boldsymbol{r}$ is one-to-one on $[a, b)$. A ***smooth*** curve C is a curve with continuously differentiable parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ such that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

§A.2 Line Integrals

Definition

A curve C is called ***simple*** if it has an injective parametrization; that is, there exists $\boldsymbol{r}: I \to \mathbb{R}^3$ such that $\boldsymbol{r}(I) = C$ and $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ implies that $x = y$. A curve C with parametrization $\boldsymbol{r}: I \to \mathbb{R}^3$ is called ***closed*** if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. A ***simple closed*** curve C is a closed curve with parametrization $\boldsymbol{r}: [a, b] \to \mathbb{R}^3$ such that $\boldsymbol{r}$ is one-to-one on $[a, b)$. A ***smooth*** curve C is a curve with continuously differentiable parametrization $\boldsymbol{r}: I \to \mathbb{R}^3$ such that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

§A.2 Line Integrals

When a parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ of curves C is mentioned, we always assume that "there is no overlap"; that is, there are no intervals $[a, b], [c, d] \subseteq I$ satisfying that $\boldsymbol{r}([a, b]) = \boldsymbol{r}([c, d])$. If in addition

1. C is a simple curve, then $\boldsymbol{r}$ is injective, or
2. C is closed, then $I = [a, b]$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$, or
3. C is simple closed, then $I = [a, b]$ and $\boldsymbol{r}$ is injective on $[a, b)$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$.
4. C is smooth, then $\boldsymbol{r}$ is continuously differentiable and $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

Theorem

Let C be a smooth curve parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$. Then

$$\ell(C) \equiv \text{the length of } C = \int_a^b \|\boldsymbol{r}'(t)\|\, dt\,.$$

§A.2 Line Integrals

When a parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ of curves C is mentioned, we always assume that "there is no overlap"; that is, there are no intervals $[a, b], [c, d] \subseteq I$ satisfying that $\boldsymbol{r}([a, b]) = \boldsymbol{r}([c, d])$. If in addition

1. C is a simple curve, then $\boldsymbol{r}$ is injective, or
2. C is closed, then $I = [a, b]$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$, or
3. C is simple closed, then $I = [a, b]$ and $\boldsymbol{r}$ is injective on $[a, b)$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$.
4. C is smooth, then $\boldsymbol{r}$ is continuously differentiable and $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

Theorem

Let C be a smooth curve parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$. Then

$$\ell(C) \equiv \text{the length of } C = \int_a^b \|\boldsymbol{r}'(t)\| \, dt \, .$$

§A.2 Line Integrals

When a parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ of curves C is mentioned, we always assume that "there is no overlap"; that is, there are no intervals $[a, b], [c, d] \subseteq I$ satisfying that $\boldsymbol{r}([a, b]) = \boldsymbol{r}([c, d])$. If in addition

1. C is a simple curve, then $\boldsymbol{r}$ is injective, or
2. C is closed, then $I = [a, b]$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$, or
3. C is simple closed, then $I = [a, b]$ and $\boldsymbol{r}$ is injective on $[a, b)$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$.
4. C is smooth, then $\boldsymbol{r}$ is continuously differentiable and $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

Theorem

Let C be a smooth curve parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$. Then

$$\ell(C) \equiv \textit{the length of } C = \int_a^b \|\boldsymbol{r}'(t)\|\, dt\,.$$

§A.2 Line Integrals

§A.2.2 Line integrals of scalar functions

In this section, we are concerned with the "integral" of a real-valued function f defined on a curve C.

Definition (Partition of curves)

Let C be a curve in space. A ***partition*** of C is a collection of curves $\{C_1, C_2, \cdots, C_n\}$ satisfying

1. $C = \bigcup_{i=1}^{n} C_i$ (so that $C_i \subseteq C$);
2. If $i \neq j$, then $C_i \cap C_j$ contains at most two points.

Let $\mathcal{P} = \{C_1, C_2, \cdots, C_n\}$ be a partition of C. The ***norm*** of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number

$$\|\mathcal{P}\| = \max\{\ell(C_1), \ell(C_2), \cdots, \ell(C_n)\},$$

where $\ell(C_j)$ denotes the length of curve C_j.

§A.2 Line Integrals

§A.2.2 Line integrals of scalar functions

In this section, we are concerned with the "integral" of a real-valued function f defined on a curve C.

Definition (Partition of curves)

Let C be a curve in space. A ***partition*** of C is a collection of curves $\{C_1, C_2, \cdots, C_n\}$ satisfying

1. $C = \bigcup_{i=1}^{n} C_i$ (so that $C_i \subseteq C$);
2. If $i \neq j$, then $C_i \cap C_j$ contains at most two points.

Let $\mathcal{P} = \{C_1, C_2, \cdots, C_n\}$ be a partition of C. The ***norm*** of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number

$$\|\mathcal{P}\| = \max\{\ell(C_1), \ell(C_2), \cdots, \ell(C_n)\},$$

where $\ell(C_j)$ denotes the length of curve C_j.

§A.2 Line Integrals

Definition (Riemann sum)

Let C be a curve in space, and $f: C \to \mathbb{R}$ is a real-valued function defined on C. A ***Riemann sum*** of f for partition $\mathcal{P}$ is a sum of the form

$$\sum_{i=1}^{n} f(q_i)\ell(C_i),$$

where $\{q_1, q_2, \cdots, q_n\}$ is a collection of points on C satisfying $q_j \in C_j$ for all $1 \leqslant j \leqslant n$.

We note that **in order to define the norm of partitions, it is required that every sub-curve** C_j **of** C **has length**. This kind of curves is called ***rectifiable*** curves, and we can only consider line integrals along rectifiable curves. In particular, a piecewise continuously differentiable curve is rectifiable.

§A.2 Line Integrals

Definition (Riemann sum)

Let C be a curve in space, and $f: C \to \mathbb{R}$ is a real-valued function defined on C. A ***Riemann sum*** of f for partition $\mathcal{P}$ is a sum of the form

$$\sum_{i=1}^{n} f(q_i)\ell(C_i)\,,$$

where $\{q_1, q_2, \cdots, q_n\}$ is a collection of points on C satisfying $q_j \in C_j$ for all $1 \leqslant j \leqslant n$.

We note that **in order to define the norm of partitions, it is required that every sub-curve C_j of C has length**. This kind of curves is called ***rectifiable*** curves, and we can only consider line integrals along rectifiable curves. In particular, a piecewise continuously differentiable curve is rectifiable.

§A.2 Line Integrals

The line integral of f along C is the limit of Riemann sums

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(q_j)\ell(C_j)$$

if the limit indeed exists. The precise definition is given below.

Definition

Let C be a rectifiable curve, and $f : C \to \mathbb{R}$ be a scalar function. The line integral of f along C is a real number L such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P} = \{C_1, C_2, \cdots, C_n\}$ is a partition of C satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to the interval $(L - \varepsilon, L + \varepsilon)$.

Whenever such an L exists, it must be unique, and the number L is denoted by $\int_C f ds$ (and when C is a closed curve, we use $\oint_C f ds$ to emphasize that the curve is closed).

§A.2 Line Integrals

The line integral of f along C is the limit of Riemann sums

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} f(q_j)\ell(C_j)$$

if the limit indeed exists. The precise definition is given below.

Definition

Let C be a rectifiable curve, and $f : C \to \mathbb{R}$ be a scalar function. The line integral of f along C is a real number L such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P} = \{C_1, C_2, \cdots, C_n\}$ is a partition of C satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to the interval $(L - \varepsilon, L + \varepsilon)$.

Whenever such an L exists, it must be unique, and the number L is denoted by $\displaystyle\int_C f\,ds$ (and when C is a closed curve, we use $\displaystyle\oint_C f\,ds$ to emphasize that the curve is closed).

§A.2 Line Integrals

The line integral of f along C is the limit of Riemann sums

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{n} f(q_j)\ell(C_j)$$

if the limit indeed exists. The precise definition is given below.

Definition

Let C be a rectifiable curve, and $f : C \to \mathbb{R}$ be a scalar function. The line integral of f along C is a real number L such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P} = \{C_1, C_2, \cdots, C_n\}$ is a partition of C satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to the interval $(L - \varepsilon, L + \varepsilon)$.

Whenever such an L exists, it must be unique, and the number L is denoted by $\int_C f ds$ (and when C is a closed curve, we use $\oint_C f ds$ to emphasize that the curve is closed).

§A.2 Line Integrals

Theorem

Let C be a (piecewise) smooth curve with (piecewise) continuously differentiable injective parametrization $\mathbf{r}:[a,b]\to\mathbb{R}^3$, and $f:C\to\mathbb{R}$ be a continuous function. Then the line integral of f along C exists and is given by

$$\int_C f\,ds = \int_a^b (f\circ\mathbf{r})(t)\big\|\mathbf{r}'(t)\big\|\,dt\,,$$

where $\big\|\mathbf{r}'(t)\big\|$ is the length of the vector $\mathbf{r}'(t)$.

§A.2 Line Integrals

Example

Evaluate $\int_C (x^2 - y + 3z)\,ds$, where C is the line segment connecting the points $(0,0,0)$ and $(1,2,1)$.

First we note that the line segment can be parameterized by

$$\boldsymbol{r}(t) = (1-t)(0,0,0) + t(1,2,1) = t\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \qquad t \in [0,1]\,.$$

Therefore, the desired line integral is given by

$$\begin{aligned}\int_C (x^2 - y + 3z)\,ds &= \int_0^1 (t^2 - 2t + 3t)\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\|\,dt \\ &= \sqrt{6}\int_0^1 (t^2 + t)\,dt = \frac{5\sqrt{6}}{6}\,.\end{aligned}$$

§A.2 Line Integrals

Example

Evaluate $\int_C (x^2 - y + 3z)\, ds$, where C is the line segment connecting the points $(0,0,0)$ and $(1,2,1)$.

First we note that the line segment can be parameterized by

$$\boldsymbol{r}(t) = (1-t)(0,0,0) + t(1,2,1) = t\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \qquad t \in [0,1]\,.$$

Therefore, the desired line integral is given by

$$\int_C (x^2 - y + 3z)\, ds = \int_0^1 (t^2 - 2t + 3t)\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\|\, dt$$
$$= \sqrt{6}\int_0^1 (t^2 + t)\, dt = \frac{5\sqrt{6}}{6}\,.$$

§A.2 Line Integrals

Example

Evaluate $\int_C (x^2 - y + 3z)\, ds$, where C is the line segment connecting the points $(0,0,0)$ and $(1,2,1)$.

First we note that the line segment can be parameterized by

$$\boldsymbol{r}(t) = (1-t)(0,0,0) + t(1,2,1) = t\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \qquad t \in [0,1]\,.$$

Therefore, the desired line integral is given by

$$\begin{aligned}\int_C (x^2 - y + 3z)\, ds &= \int_0^1 (t^2 - 2t + 3t)\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\|\, dt \\ &= \sqrt{6}\int_0^1 (t^2 + t)\, dt = \frac{5\sqrt{6}}{6}\,.\end{aligned}$$

§A.2 Line Integrals

Example

Evaluate $\int_C x\,ds$, where C is the curve starting from $(0,0)$ to $(1,1)$ along $y=x^2$ then from $(1,1)$ to $(0,0)$ along $y=x$.

Let C_1 be the piece of the curve connecting $(0,0)$ and $(1,1)$ along $y=x^2$, and C_2 be the piece of the curve connecting $(1,1)$ and $(0,0)$ along $y=x$. Then C_1 and C_2 can be parameterized by

$$\mathbf{r}_1(t)=t\mathbf{i}+t^2\mathbf{j} \quad t\in[0,1] \quad \text{and} \quad \mathbf{r}_2(t)=t\mathbf{i}+t\mathbf{j} \quad t\in[0,1],$$

respectively. Since $C=C_1\cup C_2$ and $C_1\cap C_2$ has only two points,

$$\int_C x\,ds=\int_{C_1} x\,ds+\int_{C_2} x\,ds=\int_0^1 t\|\mathbf{i}+2t\mathbf{j}\|\,dt+\int_0^1 t\|\mathbf{i}+\mathbf{j}\|\,dt$$
$$=\int_0^1\big[t\sqrt{1+4t^2}+\sqrt{2}t\big]\,dt=\frac{1}{12}(5\sqrt{5}-1)+\frac{\sqrt{2}}{2}.$$

§A.2 Line Integrals

Example

Evaluate $\int_C x\,ds$, where C is the curve starting from $(0,0)$ to $(1,1)$ along $y = x^2$ then from $(1,1)$ to $(0,0)$ along $y = x$.

Let C_1 be the piece of the curve connecting $(0,0)$ and $(1,1)$ along $y = x^2$, and C_2 be the piece of the curve connecting $(1,1)$ and $(0,0)$ along $y = x$. Then C_1 and C_2 can be parameterized by

$$\boldsymbol{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} \quad t \in [0,1] \quad \text{and} \quad \boldsymbol{r}_2(t) = t\mathbf{i} + t\mathbf{j} \quad t \in [0,1]\,,$$

respectively. Since $C = C_1 \cup C_2$ and $C_1 \cap C_2$ has only two points,

$$\int_C x\,ds = \int_{C_1} x\,ds + \int_{C_2} x\,ds = \int_0^1 t\|\mathbf{i} + 2t\mathbf{j}\|\,dt + \int_0^1 t\|\mathbf{i} + \mathbf{j}\|\,dt$$

$$= \int_0^1 \left[t\sqrt{1+4t^2} + \sqrt{2}t\right] dt = \frac{1}{12}(5\sqrt{5} - 1) + \frac{\sqrt{2}}{2}\,.$$

§A.2 Line Integrals

Example

Evaluate $\int_C x\,ds$, where C is the curve starting from $(0,0)$ to $(1,1)$ along $y=x^2$ then from $(1,1)$ to $(0,0)$ along $y=x$.

Let C_1 be the piece of the curve connecting $(0,0)$ and $(1,1)$ along $y=x^2$, and C_2 be the piece of the curve connecting $(1,1)$ and $(0,0)$ along $y=x$. Then C_1 and C_2 can be parameterized by

$$\boldsymbol{r}_1(t)=t\mathbf{i}+t^2\mathbf{j} \quad t\in[0,1] \quad \text{and} \quad \boldsymbol{r}_2(t)=t\mathbf{i}+t\mathbf{j} \quad t\in[0,1]\,,$$

respectively. Since $C=C_1\cup C_2$ and $C_1\cap C_2$ has only two points,

$$\begin{aligned}\int_C x\,ds &= \int_{C_1} x\,ds+\int_{C_2} x\,ds=\int_0^1 t\|\mathbf{i}+2t\mathbf{j}\|\,dt+\int_0^1 t\|\mathbf{i}+\mathbf{j}\|\,dt\\ &=\int_0^1\big[t\sqrt{1+4t^2}+\sqrt{2}t\big]\,dt=\frac{1}{12}(5\sqrt{5}-1)+\frac{\sqrt{2}}{2}\,.\end{aligned}$$

§A.2 Line Integrals

Example

Let C be the upper half part of the circle centered at the origin with radius $R > 0$ in the xy-plane. Evaluate the line integral $\int_C y\,ds$.

First, we parameterize C by

$$\boldsymbol{r}(t) = R\cos t\mathbf{i} + R\sin t\mathbf{j} \qquad t \in [0, \pi]\,.$$

Then

$$\begin{aligned}\int_C y\,ds &= \int_0^\pi R\sin t \|-R\sin t\mathbf{i} + R\cos t\mathbf{j}\|\,dt \\ &= \int_0^\pi R^2 \sin t\,dt = 2R^2\,.\end{aligned}$$

§A.2 Line Integrals

Example

Find the mass of a wire lying along the first octant part of the curve of intersection of the elliptic paraboloid $z = 2 - x^2 - 2y^2$ and the parabolic cylinder $z = x^2$ between $(0, 1, 0)$ and $(1, 0, 1)$ if the density of the wire at position (x, y, z) is $\varrho(x, y, z) = xy$.

Note that we can parameterize the curve C by

$$\boldsymbol{r}(t) = t\mathbf{i} + \sqrt{1 - t^2}\,\mathbf{j} + t^2\mathbf{k} \qquad t \in [0, 1] .$$

Therefore, the mass of the curve can be computed by

$$\begin{aligned}\int_C \varrho\, ds &= \int_0^1 t\sqrt{1 - t^2}\left\|\mathbf{i} + \frac{-t}{\sqrt{1 - t^2}}\mathbf{j} + 2t\mathbf{k}\right\| dt \\ &= \int_0^1 t\sqrt{2 - (1 - 2t^2)^2}\, dt = \frac{\pi}{8} + \frac{1}{4} .\end{aligned}$$

§A.2 Line Integrals

Example

Find the mass of a wire lying along the first octant part of the curve of intersection of the elliptic paraboloid $z = 2 - x^2 - 2y^2$ and the parabolic cylinder $z = x^2$ between $(0, 1, 0)$ and $(1, 0, 1)$ if the density of the wire at position (x, y, z) is $\varrho(x, y, z) = xy$.

Note that we can parameterize the curve C by

$$\boldsymbol{r}(t) = t\mathbf{i} + \sqrt{1 - t^2}\,\mathbf{j} + t^2\mathbf{k} \qquad t \in [0, 1].$$

Therefore, the mass of the curve can be computed by

$$\begin{aligned}\int_C \varrho\, ds &= \int_0^1 t\sqrt{1-t^2}\,\Big\|\mathbf{i} + \frac{-t}{\sqrt{1-t^2}}\mathbf{j} + 2t\mathbf{k}\Big\|\,dt \\ &= \int_0^1 t\sqrt{2 - (1-2t^2)^2}\,dt = \frac{\pi}{8} + \frac{1}{4}.\end{aligned}$$

§A.2 Line Integrals

Example

Find the mass of a wire lying along the first octant part of the curve of intersection of the elliptic paraboloid $z = 2 - x^2 - 2y^2$ and the parabolic cylinder $z = x^2$ between $(0, 1, 0)$ and $(1, 0, 1)$ if the density of the wire at position (x, y, z) is $\varrho(x, y, z) = xy$.

Note that we can parameterize the curve C by

$$\boldsymbol{r}(t) = t\mathbf{i} + \sqrt{1 - t^2}\,\mathbf{j} + t^2\mathbf{k} \qquad t \in [0, 1] .$$

Therefore, the mass of the curve can be computed by

$$\int_C \varrho\, ds = \int_0^1 t\sqrt{1 - t^2}\,\Big\|\mathbf{i} + \frac{-t}{\sqrt{1 - t^2}}\mathbf{j} + 2t\mathbf{k}\Big\|\,dt = \int_0^1 t\sqrt{2 - (1 - 2t^2)^2}\,dt = \frac{\pi}{8} + \frac{1}{4} .$$

§A.2 Line Integrals

§A.2.3 Line integrals of vector fields

Definition

An **oriented curve** is a curve on which a consistent tangent direction $\mathbf{T}$ is defined. In other words, an oriented curve is a (piecewise) smooth curve with a given parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ so that $\mathbf{T} \circ \boldsymbol{r} = \dfrac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$ is defined (almost everywhere).

Definition

Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth oriented curve C parameterized by $\boldsymbol{r}(t)$ for $t \in [a, b]$. The line integral of F along C is given by

$$\int_C \boldsymbol{F} \cdot \mathbf{T}\, ds\,.$$

§A.2 Line Integrals

§A.2.3 Line integrals of vector fields

Definition

An **oriented curve** is a curve on which a consistent tangent direction $\mathbf{T}$ is defined. In other words, an oriented curve is a (piecewise) smooth curve with a given parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ so that $\mathbf{T} \circ \boldsymbol{r} = \dfrac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$ is defined (almost everywhere).

Definition

Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth oriented curve C parameterized by $\boldsymbol{r}(t)$ for $t \in [a, b]$. The line integral of F along C is given by

$$\int_C \boldsymbol{F} \cdot \mathbf{T}\, ds\,.$$

§A.2 Line Integrals

§A.2.3 Line integrals of vector fields

Definition

An **oriented curve** is a curve on which a consistent tangent direction $\mathbf{T}$ is defined. In other words, an oriented curve is a (piecewise) smooth curve with a given parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ so that $\mathbf{T} \circ \boldsymbol{r} = \dfrac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$ is defined (almost everywhere).

Definition

Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth oriented curve C parameterized by $\boldsymbol{r}(t)$ for $t \in [a, b]$. The line integral of F along C is given by

$$\int_C \boldsymbol{F} \cdot \mathbf{T}\, ds\,.$$

§A.2 Line Integrals

Remark:

1. Since $\mathrm{T} \circ \boldsymbol{r} = \dfrac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$, for a curve C parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$,

$$\int_C \boldsymbol{F} \cdot \mathrm{T}\, ds = \int_a^b (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} \|\boldsymbol{r}'(t)\|\, dt = \int_a^b (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt.$$

   Since $\boldsymbol{r}'(t)\, dt = d\boldsymbol{r}(t)$, sometimes we also use $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ to denote the line integral of $\boldsymbol{F}$ along the oriented curve C parameterized by $\boldsymbol{r}$.

2. Given an oriented curve C and $\boldsymbol{F} : C \to \mathbb{R}^3$, we sometimes use $\displaystyle\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r}$ to denote the line integral $\displaystyle\int_C \boldsymbol{F} \cdot (-\mathrm{T})\, ds$, where $-\mathrm{T}$ is the tangent direction opposite to the orientation of C.

§A.2 Line Integrals

Remark:

1. Since $\mathbf{T} \circ \boldsymbol{r} = \dfrac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|}$, for a curve C parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$,

$$\int_C \boldsymbol{F} \cdot \mathbf{T}\, ds = \int_a^b (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} \|\boldsymbol{r}'(t)\|\, dt = \int_a^b (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt.$$

Since $\boldsymbol{r}'(t)\, dt = d\boldsymbol{r}(t)$, sometimes we also use $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ to denote the line integral of $\boldsymbol{F}$ along the oriented curve C parameterized by $\boldsymbol{r}$.

2. Given an oriented curve C and $\boldsymbol{F} : C \to \mathbb{R}^3$, we sometimes use $\displaystyle\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r}$ to denote the line integral $\displaystyle\int_C \boldsymbol{F} \cdot (-\mathbf{T})\, ds$, where $-\mathbf{T}$ is the tangent direction opposite to the orientation of C.

§A.2 Line Integrals

3. Let C be a smooth oriented curve parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ and $\boldsymbol{F} : C \to \mathbb{R}^3$. Then $-C$, the oriented curve with opposite orientation w.r.t. C, can be parameterized by $\boldsymbol{r}_1 : [-b, -a] \to \mathbb{R}^3$ given by $\boldsymbol{r}_1(t) = \boldsymbol{r}(-t)$ so that

$$\begin{aligned}\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{-b}^{-a} (\boldsymbol{F} \circ \boldsymbol{r}_1)(t) \cdot \boldsymbol{r}_1'(t) dt \\ &= \int_{-b}^{-a} (\boldsymbol{F} \circ \boldsymbol{r})(-t) \cdot (-\boldsymbol{r}')(-t)\, dt \\ &= \int_{b}^{a} (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt \\ &= -\int_{a}^{b} (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt = \int_C \boldsymbol{F} \cdot (-\mathrm{T})\, ds\,.\end{aligned}$$

This explains $\displaystyle\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot (-\mathrm{T})\, ds$.

§A.2 Line Integrals

3. Let C be a smooth oriented curve parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ and $\boldsymbol{F} : C \to \mathbb{R}^3$. Then $-C$, the oriented curve with opposite orientation w.r.t. C, can be parameterized by $\boldsymbol{r}_1 : [-b, -a] \to \mathbb{R}^3$ given by $\boldsymbol{r}_1(t) = \boldsymbol{r}(-t)$ so that

$$\begin{aligned}
\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{-b}^{-a} (\boldsymbol{F} \circ \boldsymbol{r}_1)(t) \cdot \boldsymbol{r}_1'(t) dt \\
&= \int_{-b}^{-a} (\boldsymbol{F} \circ \boldsymbol{r})(-t) \cdot (-\boldsymbol{r}')(-t)\, dt \\
&= \int_{b}^{a} (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt \\
&= -\int_{a}^{b} (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt = \int_C \boldsymbol{F} \cdot (-\mathbf{T})\, ds\,.
\end{aligned}$$

This explains $\displaystyle\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot (-\mathbf{T})\, ds.$

§A.2 Line Integrals

3. Let C be a smooth oriented curve parameterized by $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ and $\boldsymbol{F} : C \to \mathbb{R}^3$. Then $-C$, the oriented curve with opposite orientation w.r.t. C, can be parameterized by $\boldsymbol{r}_1 : [-b, -a] \to \mathbb{R}^3$ given by $\boldsymbol{r}_1(t) = \boldsymbol{r}(-t)$ so that

$$\begin{aligned}\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_{-b}^{-a} (\boldsymbol{F} \circ \boldsymbol{r}_1)(t) \cdot \boldsymbol{r}_1'(t) dt \\ &= \int_{-b}^{-a} (\boldsymbol{F} \circ \boldsymbol{r})(-t) \cdot (-\boldsymbol{r}')(-t)\, dt \\ &= \int_{b}^{a} (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt \\ &= -\int_{a}^{b} (\boldsymbol{F} \circ \boldsymbol{r})(t) \cdot \boldsymbol{r}'(t)\, dt = \int_C \boldsymbol{F} \cdot (-\mathbf{T})\, ds\,.\end{aligned}$$

This explains $\displaystyle\int_{-C} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot (-\mathbf{T})\, ds$.

§A.2 Line Integrals

Example

Find the work done by the force field

$$\boldsymbol{F}(x, y, z) = -\frac{1}{2}x\mathbf{i} - \frac{1}{2}y\mathbf{j} + \frac{1}{4}\mathbf{k}$$

on a particle as it moves along the helix parameterized by

$$\boldsymbol{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

from the point $(1, 0, 0)$ to the point $(-1, 0, 3\pi)$. Note that such a helix is parameterized by $\boldsymbol{r}(t)$ with $t \in [0, 3\pi]$. Therefore,

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_0^{3\pi} \Big(-\frac{1}{2}\cos t\mathbf{i} - \frac{1}{2}\sin t\mathbf{j} + \frac{1}{4}\mathbf{k}\Big) \cdot \big(-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}\big)\,dt \\ &= \int_0^{3\pi} \Big(\frac{1}{2}\sin t\cos t - \frac{1}{2}\sin t\cos t + \frac{1}{4}\Big)\,dt = \frac{3\pi}{4}.\end{aligned}$$

§A.2 Line Integrals

Example

Let $\boldsymbol{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ from $(0,0)$ to $(1,1)$ along

1. the straight line $y = x$,
2. the curve $y = x^2$, and
3. the piecewise smooth path consisting of the straight line segments from $(0,0)$ to $(0,1)$ and from $(0,1)$ to $(1,1)$.

For the straight line case, we parameterize the path by $\boldsymbol{r}(t) = (t, t)$ for $t \in [0,1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^2\mathbf{i} + 2t^2\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})dt = \int_0^1 3t^2 dt = 1\,.$$

§A.2 Line Integrals

Example

Let $\boldsymbol{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ from $(0, 0)$ to $(1, 1)$ along

1. the straight line $y = x$,
2. the curve $y = x^2$, and
3. the piecewise smooth path consisting of the straight line segments from $(0, 0)$ to $(0, 1)$ and from $(0, 1)$ to $(1, 1)$.

For the straight line case, we parameterize the path by $\boldsymbol{r}(t) = (t, t)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^2\mathbf{i} + 2t^2\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) dt = \int_0^1 3t^2 dt = 1\,.$$

§A.2 Line Integrals

Example (cont.)

For the case of parabola, we parameterize the path by $\boldsymbol{r}(t) = (t, t^2)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^4\mathbf{i} + 2t^3\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})dt = \int_0^1 5t^4 dt = 1\,.$$

For the piecewise linear case, we let C_1 denote the line segment joining $(0,0)$ and $(0,1)$, and let C_2 denote the line segment joining $(0,1)$ and $(1,1)$. Note that we can parameterize C_1 and C_2 by

$$\boldsymbol{r}_1(t) = t\mathbf{j} \quad t \in [0,1] \quad \text{and} \quad \boldsymbol{r}_2(t) = t\mathbf{i} + \mathbf{j} \quad t \in [0,1]\,,$$

respectively. Therefore,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 t^2\mathbf{i} \cdot \mathbf{j}\, dt + \int_0^1 (\mathbf{i} + 2t\mathbf{j}) \cdot \mathbf{i}\, dt = 1\,.$$

§A.2 Line Integrals

Example (cont.)

For the case of parabola, we parameterize the path by $\boldsymbol{r}(t) = (t, t^2)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (t^4\mathbf{i} + 2t^3\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})\,dt = \int_0^1 5t^4\,dt = 1\,.$$

For the piecewise linear case, we let C_1 denote the line segment joining $(0,0)$ and $(0,1)$, and let C_2 denote the line segment joining $(0,1)$ and $(1,1)$. Note that we can parameterize C_1 and C_2 by

$$\boldsymbol{r}_1(t) = t\mathbf{j} \quad t \in [0,1] \quad \text{and} \quad \boldsymbol{r}_2(t) = t\mathbf{i} + \mathbf{j} \quad t \in [0,1]\,,$$

respectively. Therefore,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 t^2\mathbf{i} \cdot \mathbf{j}\,dt + \int_0^1 (\mathbf{i} + 2t\mathbf{j}) \cdot \mathbf{i}\,dt = 1\,.$$

§A.2 Line Integrals

Example

Let $\boldsymbol{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ from $(1, 0)$ to $(0, -1)$ along

1. the straight line segment joining these points, and
2. three-quarters of the circle of unit radius centered at the origin and traversed counter-clockwise.

For the first case, we parameterize the path by $\boldsymbol{r}(t) = (1 - t, -t)$ for $t \in [0, 1]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \big[- t\mathbf{i} + (t - 1)\mathbf{j}\big] \cdot (-\mathbf{i} - \mathbf{j})\, dt = 1\,.$$

For the second case, we parameterize the path by $\boldsymbol{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ for $t \in \big[0, \frac{3\pi}{2}\big]$. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{\frac{3\pi}{2}} (\sin t\mathbf{i} - \cos t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})\, dt = -\frac{3\pi}{2}\,.$$

§A.3 Green's Theorem

Let $R \subseteq \mathbb{R}^2$ be a region enclosed by a simply closed curve C and $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$ be a vector fields on (an open set containing) R, where C is ***oriented counterclockwise*** so that

C is traversed once so that the region R always lies to the left.

The line integral of $\boldsymbol{F}$ along an oriented curve C sometimes is written as

$$\oint_C Mdx + Ndy$$

since symbolically we have $d\boldsymbol{r} = dx\mathbf{i} + dy\mathbf{j}$ so that

$$\boldsymbol{F} \cdot d\boldsymbol{r} = (M\mathbf{i} + N\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = Mdx + Ndy\,.$$

The right-hand side of the identity above is called a ***differential form***.

§A.3 Green's Theorem

Theorem (Green's Theorem)

Let R be a plane region enclosed by a closed curve C oriented counterclockwise; that is, C is traversed once so that the region R always lies to the left. If M and N have continuous first partial derivatives in an open region containing R, then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)(x, y)\, dA\,.$$

Remark: If $\boldsymbol{F}$ is a two-dimensional vector field given by $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$, then under the assumption of Green's Theorem,

$$\oint_C \boldsymbol{F} \cdot \mathbf{T}\, ds = \iint_R (\mathrm{curl}\boldsymbol{F})(x, y)\, dA\,.$$

This is sometimes called **Green's Theorem in Tangential Form**.

§A.3 Green's Theorem

Theorem (Green's Theorem)

Let R be a plane region enclosed by a closed curve C oriented counterclockwise; that is, C is traversed once so that the region R always lies to the left. If M and N have continuous first partial derivatives in an open region containing R, then

$$\oint_C Mdx + Ndy = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)(x, y)\, dA\,.$$

Remark: If $\boldsymbol{F}$ is a two-dimensional vector field given by $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$, then under the assumption of Green's Theorem,

$$\oint_C \boldsymbol{F} \cdot \mathbf{T}\, ds = \iint_R (\mathrm{curl}\boldsymbol{F})(x, y)\, dA\,.$$

This is sometimes called **Green's Theorem in Tangential Form**.

§A.3 Green's Theorem

Remark: Let R be a region enclosed by a smooth simply closed curve C with **outward-pointing** unit normal $\mathbf{N}$ on C, and $\boldsymbol{F}$ be a smooth vector field defined on an open region containing R. We are interested in $\displaystyle\oint_C \boldsymbol{F}\cdot\mathbf{N}\,ds$, the line integral of $\boldsymbol{F}\cdot\mathbf{N}$ along C.

Suppose that $\boldsymbol{F} = M\mathbf{i} + N\mathbf{j}$, and C is parameterized by $\boldsymbol{r}(t) = x(t)\mathbf{i}+y(t)\mathbf{j}$, $t\in[a,b]$, so that C is oriented counterclockwise. Define $\boldsymbol{G} = -N\mathbf{i} + M\mathbf{j}$. Then Green's Theorem (in tangential form) implies that

$$\oint_C -N\,dx + M\,dy = \oint_C \boldsymbol{G}\cdot d\boldsymbol{r} = \iint_R \operatorname{curl}\boldsymbol{G}\,dA = \iint_R (M_x + N_y)\,dA$$
$$= \iint_R \operatorname{div}\boldsymbol{F}\,dA\,.$$

§A.3 Green's Theorem

On the other hand, if $\boldsymbol{r}$ is a counterclockwise parametrization of C, then

$$\mathbf{N}(\boldsymbol{r}(t)) = \frac{y'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{i} - \frac{x'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{j} \qquad \forall\, t \in [a, b]\,;$$

thus

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot\mathbf{N}\,ds &= \int_a^b (\boldsymbol{F}\cdot\mathbf{N})(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\,dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot\mathbf{N}(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\,dt \\ &= \int_a^b \big[M(x(t),y(t))y'(t) - N(x(t),y(t))x'(t)\big]\,dt \\ &= \oint_C -N\,dx + M\,dy = \oint_C \boldsymbol{G}\cdot d\boldsymbol{r} = \iint_R \mathrm{div}\boldsymbol{F}\,dA\,.\end{aligned}$$

Therefore,

$$\oint_C \boldsymbol{F}\cdot\mathbf{N}\,ds = \iint_R \mathrm{div}\boldsymbol{F}\,dA\,.$$

This is sometimes called **Green's Theorem in Normal Form**.

§A.3 Green's Theorem

On the other hand, if $\boldsymbol{r}$ is a counterclockwise parametrization of C, then

$$\mathrm{N}(\boldsymbol{r}(t)) = \frac{y'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{i} - \frac{x'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{j} \qquad \forall\, t \in [a, b]\,;$$

thus

$$\oint_C \boldsymbol{F}\cdot \mathrm{N}\, ds = \int_a^b (\boldsymbol{F}\cdot\mathrm{N})(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\, dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot \mathrm{N}(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\, dt$$

$$= \int_a^b \big[M(x(t), y(t))y'(t) - N(x(t), y(t))x'(t)\big]\, dt$$

$$= \oint_C -N\, dx + M\, dy = \oint_C \boldsymbol{G}\cdot d\boldsymbol{r} = \iint_R \mathrm{div}\boldsymbol{F}\, dA\,.$$

Therefore,

$$\oint_C \boldsymbol{F}\cdot\mathrm{N}\, ds = \iint_R \mathrm{div}\boldsymbol{F}\, dA\,.$$

This is sometimes called **Green's Theorem in Normal Form**.

§A.3 Green's Theorem

On the other hand, if $\boldsymbol{r}$ is a counterclockwise parametrization of C, then

$$\mathrm{N}(\boldsymbol{r}(t)) = \frac{y'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{i} - \frac{x'(t)}{\|\boldsymbol{r}'(t)\|}\mathbf{j} \qquad \forall\, t \in [a, b];$$

thus

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot\mathrm{N}\,ds &= \int_a^b (\boldsymbol{F}\cdot\mathrm{N})(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\,dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t))\cdot\mathrm{N}(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|\,dt \\ &= \int_a^b \big[M(x(t),y(t))y'(t) - N(x(t),y(t))x'(t)\big]\,dt \\ &= \oint_C -N\,dx + M\,dy = \oint_C \boldsymbol{G}\cdot d\boldsymbol{r} = \iint_R \mathrm{div}\,\boldsymbol{F}\,dA\,.\end{aligned}$$

Therefore,

$$\oint_C \boldsymbol{F}\cdot\mathrm{N}\,ds = \iint_R \mathrm{div}\,\boldsymbol{F}\,dA\,.$$

This is sometimes called **Green's Theorem in Normal Form**.

§A.3 Green's Theorem

Example

Use Green's Theorem to evaluate the line integral

$$\oint_C y^3 dx + (x^3 + 3xy^2) dy,$$

where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$.

Let $R = \{(x,y) \mid 0 \leqslant x \leqslant 1, x^3 \leqslant y \leqslant x\}$. Then Green's Theorem implies that

$$\oint_C y^3 dx + (x^3 + 3xy^2) dy = \iint_R \Big[\frac{\partial}{\partial x}(x^3 + 3xy^2) - \frac{\partial}{\partial y} y^3\Big]\, dA$$
$$= \iint_R 3x^2\, dA = \int_0^1 \Big(\int_{x^3}^x 3x^2\, dy\Big)\, dx = \frac{1}{4}.$$

§A.3 Green's Theorem

Example

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be the annular region $\mathcal{D} = \{(x, y) \,|\, 1 < x^2 + y^2 < 4\}$, $\boldsymbol{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$, and $C \subseteq \mathcal{D}$ be a simple closed curve oriented counterclockwise so that the origin is inside the region enclosed by C. Find $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Choose $r > 1$ so that the circle centered at the origin with radius r lies in the intersection of $\mathcal{D}$ and the finite region enclosed by C. Let C_r denote this circle with clockwise orientation, and pick a line segment B connecting C and C_r (with starting point on C and end-point on C_r). Define Γ as the oriented curve $B \cup C_r \cup (-B) \cup C$, where $-B$ denotes oriented curve B with opposite orientation, and let R be the region enclosed by Γ. Then $R \subseteq \mathcal{D}$ and R is the region lies to the left of Γ.

§A.3 Green's Theorem

Example

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be the annular region $\mathcal{D} = \{(x, y) \,|\, 1 < x^2 + y^2 < 4\}$, $\boldsymbol{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$, and $C \subseteq \mathcal{D}$ be a simple closed curve oriented counterclockwise so that the origin is inside the region enclosed by C. Find $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Choose $r > 1$ so that the circle centered at the origin with radius r lies in the intersection of $\mathcal{D}$ and the finite region enclosed by C. Let C_r denote this circle with clockwise orientation, and pick a line segment B connecting C and C_r (with starting point on C and end-point on C_r). Define Γ as the oriented curve $B \cup C_r \cup (-B) \cup C$, where $-B$ denotes oriented curve B with opposite orientation, and let R be the region enclosed by Γ. Then $R \subseteq \mathcal{D}$ and R is the region lies to the left of Γ.

§A.3 Green's Theorem

Example

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be the annular region $\mathcal{D} = \left\{(x, y) \,\middle|\, 1 < x^2 + y^2 < 4\right\}$, $\boldsymbol{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$, and $C \subseteq \mathcal{D}$ be a simple closed curve oriented counterclockwise so that the origin is inside the region enclosed by C. Find $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

Choose $r > 1$ so that the circle centered at the origin with radius r lies in the intersection of $\mathcal{D}$ and the finite region enclosed by C. Let C_r denote this circle with clockwise orientation, and pick a line segment B connecting C and C_r (with starting point on C and end-point on C_r). Define Γ as the oriented curve $B \cup C_r \cup (-B) \cup C$, where $-B$ denotes oriented curve B with opposite orientation, and let R be the region enclosed by Γ. Then $R \subseteq \mathcal{D}$ and R is the region lies to the left of Γ.

§A.3 Green's Theorem

Example (cont.)

Therefore, Green's Theorem implies that

$$\int_{\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_{R} \mathrm{curl}\boldsymbol{F}\, dA = 0\,.$$

On the other hand,

$$\int_{\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{B} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{-B} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C} \boldsymbol{F} \cdot d\boldsymbol{r}\,;$$

thus by the fact that $\displaystyle\int_{-B} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{B} \boldsymbol{F} \cdot d\boldsymbol{r}$, we conclude that

$$\int_{C} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\Gamma} \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

or equivalently,

$$\int_{C} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{C_r} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{-C_r} \boldsymbol{F} \cdot d\boldsymbol{r}\,.$$

§A.3 Green's Theorem

Example (cont.)

In other words, the line integral of $\boldsymbol{F}$ along C is the same as the line integral of $\boldsymbol{F}$ along the circle C_r with counterclockwise orientation. Since $-C_r$ can be parameterized by

$$\boldsymbol{r}(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j} \qquad t \in [0, 2\pi]\,,$$

we find that

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi} \Big(\frac{r\sin t}{r^2}\mathbf{i} - \frac{r\cos t}{r^2}\mathbf{j}\Big)\cdot\big(-r\sin t\mathbf{i} + r\cos t\mathbf{j}\big)\,dt \\ &= \int_0^{2\pi} (-1)\,dt = -2\pi\,.\end{aligned}$$

§A.4 The Surface Integrals

§A.4.1 Parametric surfaces

Definition (Parametric Surfaces)

Let X, Y and Z be functions of u and v that are continuous on a domain D in the uv-plane. The collection of points

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}$$

is called a parametric surface. The equations $x = X(u,v)$, $y = Y(u,v)$, and $z = Z(u,v)$ are the parametric equations for the surface, and $\boldsymbol{r} : D \to \mathbb{R}^3$ given by $\boldsymbol{r}(u,v) = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k}$ is called a parametrization of Σ.

§A.4 The Surface Integrals

Definition (Regular Surfaces)

A parametric surface

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}$$

is said to be ***regular*** if X, Y, Z are continuously differentiable functions and

$$\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v) \neq \mathbf{0} \qquad \forall\, (u,v) \in D\,,$$

where

$$\boldsymbol{r}_u(u,v) \equiv X_u(u,v)\mathbf{i} + Y_u(u,v)\mathbf{j} + Z_u(u,v)\mathbf{k}\,,$$
$$\boldsymbol{r}_v(u,v) \equiv X_v(u,v)\mathbf{i} + Y_v(u,v)\mathbf{j} + Z_v(u,v)\mathbf{k}\,.$$

§A.4 The Surface Integrals

Example

Let R be an open region in the plane, and $f : R \to \mathbb{R}$ be a continuous function. Then the graph of f is a parametric surface. In fact,

the graph of $f = \Big\{ \boldsymbol{r} \in \mathbb{R}^3 \,\Big|\, \boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k} \Big)$ for some $(x, y) \in R \Big\}$.

Therefore, a parametric surface can be viewed as a generalization of surfaces being graphs of functions.

Example

Let $\mathbb{S}^2 = \big\{ (x, y, z) \in \mathbb{R}^3 \,\big|\, x^2 + y^2 + z^2 = 1 \big\}$ be the unit sphere in $\mathbb{R}^3$. Consider

$$\boldsymbol{r}(\theta, \phi) = \cos\theta \sin\phi\,\mathbf{i} + \sin\theta \sin\phi\,\mathbf{j} + \cos\phi\,\mathbf{k}\,,$$

where $(\theta, \phi) \in D = [0, 2\pi) \times [0, \pi)$. Then $\boldsymbol{r} : D \to \mathbb{S}^2$ is a continuous bijection; thus $\mathbb{S}^2$ is a parametric surface.

§A.4 The Surface Integrals

Example

Let R be an open region in the plane, and $f : R \to \mathbb{R}$ be a continuous function. Then the graph of f is a parametric surface. In fact,

the graph of $f = \Big\{ \boldsymbol{r} \in \mathbb{R}^3 \,\Big|\, \boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}\Big)$ for some $(x, y) \in R \Big\}$.

Therefore, a parametric surface can be viewed as a generalization of surfaces being graphs of functions.

Example

Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Consider

$$\boldsymbol{r}(\theta, \phi) = \cos\theta \sin\phi \mathbf{i} + \sin\theta \sin\phi \mathbf{j} + \cos\phi \mathbf{k} \,,$$

where $(\theta, \phi) \in D = [0, 2\pi) \times [0, \pi)$. Then $\boldsymbol{r} : D \to \mathbb{S}^2$ is a continuous bijection; thus $\mathbb{S}^2$ is a parametric surface.

§A.4 The Surface Integrals

Example

Consider the torus shown below

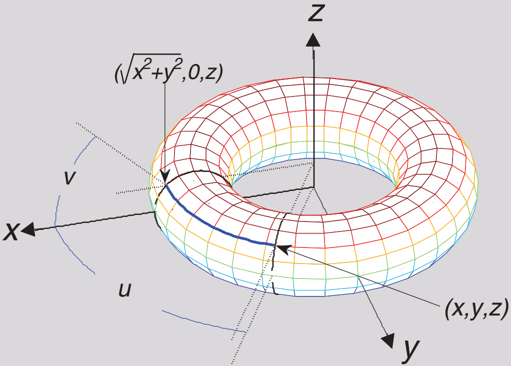


Figure 1: Torus with parametrization $\boldsymbol{r}(u, v)$. (temporary picture)

§A.4 The Surface Integrals

Example (cont.)

Note that the torus has a parametrization

$$\boldsymbol{r}(u,v) = (a + b\cos v)\cos u\,\mathbf{i} + (a + b\cos v)\sin u\,\mathbf{j} + b\sin v\,\mathbf{k}\,,$$

where $(u,v) \in [0,2\pi) \times [0,2\pi)$. Therefore, the torus is a parametric surface.

§A.4 The Surface Integrals

§A.4.2 Surface area of parametric surfaces

Theorem

Let D be an open region in the plane, and

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}$$

be a regular parametric surface so that r is continuously differentiable; that is, X_u, X_v, Y_u, Y_v, Z_u, Z_v are continuous. Then

$$\text{the surface area of } \Sigma = \iint_D \left\| \boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v) \right\| d(u,v) \,.$$

§A.4 The Surface Integrals

Example

The theorem above provides one specific way of evaluating the surface integrals: if the surface Σ is in fact a subset of the graph of a function $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$; that is, $\Sigma \subseteq \left\{x, y, f(x, y)) \,\middle|\, (x, y) \in R\right\}$, then Σ has a parametrization

$$\boldsymbol{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}\,, \qquad (x, y) \in R\,.$$

Then

$$\|\boldsymbol{r}_x(x, y) \times \boldsymbol{r}_y(x, y)\|_{\mathbb{R}^3}^2 = 1 + \left|\frac{\partial f}{\partial x}(x, y)\right|^2 + \left|\frac{\partial f}{\partial y}(x, y)\right|^2;$$

thus

$$\text{the surface area of } \Sigma = \iint_R \sqrt{1 + \left|\frac{\partial f}{\partial x}(x, y)\right|^2 + \left|\frac{\partial f}{\partial y}(x, y)\right|^2}\, dA\,.$$

§A.4 The Surface Integrals

Example

Given the parametrization of the unit sphere $\mathbb{S}^2$

$$\boldsymbol{r}(\theta,\phi)=\cos\theta\sin\phi\,\mathbf{i}+\sin\theta\sin\phi\,\mathbf{j}+\cos\phi\,\mathbf{k}\,,\ \ (\theta,\phi)\in[0,2\pi]\times[0,\pi]\,,$$

we find that

$$\begin{aligned}\boldsymbol{r}_\theta(\theta,\phi)&=-\sin\theta\sin\phi\,\mathbf{i}+\cos\theta\sin\phi\,\mathbf{j}\,,\\ \boldsymbol{r}_\phi(\theta,\phi)&=\cos\theta\cos\phi\,\mathbf{i}+\sin\theta\cos\phi\,\mathbf{j}-\sin\phi\,\mathbf{k}\end{aligned}$$

so that

$$\begin{aligned}(\boldsymbol{r}_\theta\times\boldsymbol{r}_\phi)(\theta,\phi)&=-\cos\theta\sin^2\phi\,\mathbf{i}-\sin\theta\sin^2\phi\,\mathbf{j}-\sin\phi\cos\phi\,\mathbf{k}\\ &=-\sin\phi\big(\cos\theta\sin\phi\,\mathbf{i}+\sin\theta\sin\phi\,\mathbf{j}+\cos\phi\,\mathbf{k}\big)\,.\end{aligned}$$

Therefore, the surface area of $\mathbb{S}^2$ is

$$\iint_{[0,2\pi]\times[0,\pi]}\big\|(\boldsymbol{r}_\theta\times\boldsymbol{r}_\phi)(\theta,\phi)\big\|\,d(\theta,\phi)=\int_0^\pi\Big(\int_0^{2\pi}\sin\phi\,d\theta\Big)d\phi=4\pi\,.$$

§A.4 The Surface Integrals

Example

Given the parametrization of the torus given in previous example by

$$\boldsymbol{r}(u,v) = (a+b\cos v)\cos u\,\mathbf{i} + (a+b\cos v)\sin u\,\mathbf{j} + b\sin v\,\mathbf{k}\,,$$

where $(u,v) \in [0,2\pi)\times[0,2\pi)$, we find that

$$\begin{aligned}\boldsymbol{r}_u(u,v) &= -(a+b\cos v)\sin u\,\mathbf{i} + (a+b\cos v)\cos u\,\mathbf{j}\,,\\ \boldsymbol{r}_v(u,v) &= -b\sin v\cos u\,\mathbf{i} - b\sin v\sin u\,\mathbf{j} + b\cos v\,\mathbf{k}\,;\end{aligned}$$

thus

$$(\boldsymbol{r}_u\times\boldsymbol{r}_v)(u,v) = b(a+b\cos v)\big(\cos u\cos v\mathbf{i} + \sin u\cos v\mathbf{j} + \sin v\mathbf{k}\big)\,.$$

Therefore, the surface area of the torus is

$$\begin{aligned}\iint_{[0,2\pi]\times[0,2\pi]} b(a+b\cos v)\,d(u,v) &= \int_0^{2\pi}\Big(\int_0^{2\pi}(ab+b^2\cos v)\,du\Big)dv\\ &= 4\pi^2 ab\,.\end{aligned}$$

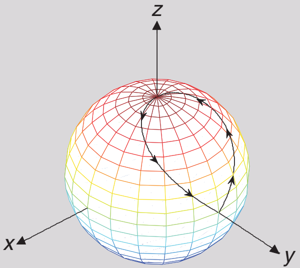
§A.4 The Surface Integrals

Example

Let C be a smooth curve parameterized by

$$\boldsymbol{r}(t) = (\cos t \sin t, \sin t \sin t, \cos t), \qquad t \in \mathrm{I} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Then clearly C is on the unit sphere $\mathbb{S}^2$ since $\|\boldsymbol{r}(t)\|_{\mathbb{R}^3} = 1$ for all $t \in \mathrm{I}$. Since C is a closed curve, C divides $\mathbb{S}^2$ into two parts. Find the surface area of the part Σ "enclosed" by C.



§A.4 The Surface Integrals

Example (cont.)

To compute the surface area of Σ, we need to find a way to parameterize Σ. Naturally we try to parameterize Σ using the spherical coordinate. In other words, let $\mathrm{R} = (0, 2\pi) \times (0, \pi)$ and $\boldsymbol{\psi} : \mathrm{R} \to \mathbb{R}^3$ be defined by

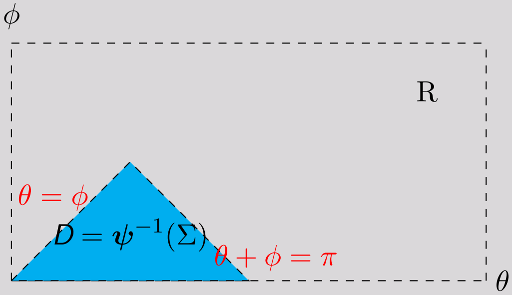
$$\boldsymbol{\psi}(\theta, \phi) = \cos\theta \sin\phi \mathbf{i} + \sin\theta \sin\phi \mathbf{j} + \cos\phi \mathbf{k} \,,$$

and we would like to find a region $D \subseteq \mathrm{R}$ such that $\boldsymbol{\psi}(D) = \Sigma$. Suppose that $\gamma(t) = \big(\theta(t), \phi(t)\big)$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is a curve in R such that $(\boldsymbol{\psi} \circ \gamma)(t) = \boldsymbol{r}(t)$. Then for $t \in \left[0, \frac{\pi}{2}\right]$, the identity $\cos t = \cos\phi(t)$ implies that $\phi(t) = t$; thus the identities $\cos t \sin t = \cos\theta(t) \sin\phi(t)$ and $\sin t \sin t = \sin\theta(t) \sin\phi(t)$ further imply that $\theta(t) = t$.

§A.4 The Surface Integrals

Example (cont.)

On the other hand, for $t \in \left[-\frac{\pi}{2}, 0\right]$, the identity $\cos t = \cos \phi(t)$, where $\phi(t) \in (0, \pi)$, implies that $\phi(t) = -t$; thus the identities $\cos t \sin t = \cos \theta(t) \sin \phi(t)$ and $\sin t \sin t = \sin \theta(t) \sin \phi(t)$ further imply that $\theta(t) = \pi + t$.



§A.4 The Surface Integrals

Example (cont.)

Since

$$\psi_\theta(\theta,\phi) = -\sin\theta\sin\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j}$$
$$\psi_\phi(\theta,\phi) = \cos\theta\cos\phi\,\mathbf{i} + \sin\theta\cos\phi\,\mathbf{j} - \sin\phi\,\mathbf{k}$$

we find that

$$\begin{aligned}&\big\|(\psi_\theta\times\psi_\phi)(\theta,\phi)\big\|^2\\ &= \big\|-\cos\theta\sin^2\phi\,\mathbf{i} - \sin\theta\sin^2\phi\,\mathbf{j} - (\sin^2\theta+\cos^2\theta)\sin\phi\cos\phi\,\mathbf{k}\big\|^2\\ &= (\cos^2\theta+\sin^2\theta)\sin^4\phi + \sin^2\phi\cos^2\phi = \sin^2\phi\,,\end{aligned}$$

the area of the desired surface can be computed by

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_\phi^{\pi-\phi}\sin\phi\,d\theta d\phi &= \int_0^{\frac{\pi}{2}}(\pi-2\phi)\sin\phi\,d\phi\\ &= \big(-\pi\cos\phi + 2\phi\cos\phi - 2\sin\phi\big)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} = \pi - 2\,.\end{aligned}$$

§A.4 The Surface Integrals

Example (cont.)

Another way to parameterize Σ is to view Σ as the graph of function $z = \sqrt{1 - x^2 - y^2}$ over D, where D is the projection of Σ along z-axis onto xy-plane. We note that the boundary of D can be parameterized by

$$\widetilde{\boldsymbol{r}}(t) = \cos t \sin t\, \mathbf{i} + \sin t \sin t\, \mathbf{j}\,, \qquad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Let $(x, y) \in \partial D$. Then $x^2 + y^2 = y$; thus Σ can also be parameterized by $\psi : D \to \mathbb{R}^3$, where

$$\psi(x, y) = x\mathbf{i} + y\mathbf{j} + \sqrt{1 - x^2 - y^2}\mathbf{k} \quad \text{and} \quad D = \left\{(x, y) \,\middle|\, x^2 + y^2 \leqslant y\right\}.$$

§A.4 The Surface Integrals

Example (cont.)

Therefore, with f denoting the function $f(x, y) = \sqrt{1 - x^2 - y^2}$, the surface area of Σ is

$$\int_D \sqrt{1 + f_x^2 + f_y^2}\, dA = \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \frac{1}{\sqrt{1 - x^2 - y^2}}\, dxdy$$

$$= \int_0^1 \arcsin \frac{x}{\sqrt{1 - y^2}} \Big|_{x=-\sqrt{y-y^2}}^{x=\sqrt{y-y^2}} dy = 2 \int_0^1 \arcsin \frac{\sqrt{y}}{\sqrt{1 + y}}\, dy;$$

thus making a change of variable $y = \tan^2 \theta$ we conclude that the surface area of Σ is

$$2 \int_0^{\frac{\pi}{4}} \arcsin \frac{\tan \theta}{\sec \theta}\, d(\tan^2 \theta) = 2 \int_0^{\frac{\pi}{4}} \theta\, d\big(\tan^2 \theta\big)$$

$$= 2 \Big[\theta \tan^2 \theta \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2 \theta\, d\theta \Big] = \pi - 2\,.$$

§A.4 The Surface Integrals

§A.4.3 Surface integrals of scalar functions

Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface and $f : \Sigma \to \mathbb{R}$ be a real-valued function. We partition Σ into small pieces $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ so that $\Sigma_i \cap \Sigma_j$ has zero area if $i \neq j$ and $\Sigma = \bigcup_{k=1}^{n} \Sigma_k$. A Riemann sum of f for partition $\{\Sigma_1, \cdots, \Sigma_n\}$ (of Σ) takes the form

$$\sum_{k=1}^{n} f(p_k)\sigma(\Sigma_k),$$

where $p_1, \cdots, p_n$ are points on Σ satisfying $p_k \in \Sigma_k$, and $\sigma(\Sigma_k)$ denotes the surface area of Σ_k. The limit of Riemann sums as $\max\{\text{diam}(\Sigma_1), \text{diam}(\Sigma_2), \cdots, \text{diam}(\Sigma_n)\}$ approaches zero, if exists, is called the surface integral of f on Σ, and is denoted by $\int_\Sigma f\,dS$.

§A.4 The Surface Integrals

Theorem

Let D be an open region in the plane, and

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}$$

be a regular parametric surface so that $\boldsymbol{r}$ is continuously differentiable, and $f : \Sigma \to \mathbb{R}$ be a continuous function. Then the surface integral of f on Σ exists and is given by

$$\iint_D (f \circ \boldsymbol{r})(u,v) \big\| (\boldsymbol{r}_u \times \boldsymbol{r}_v)(u,v) \big\| \, d(u,v) \,.$$

Remark: If the surface Σ is the graph of a function $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$, then for a continuous function $g : \Sigma \to \mathbb{R}$, we have

$$\int_\Sigma g \, dS = \iint_R g(x,y,f(x,y)) \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2} \, dA \,.$$

§A.4 The Surface Integrals

Theorem

Let D be an open region in the plane, and

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}$$

be a regular parametric surface so that $\boldsymbol{r}$ is continuously differentiable, and $f : \Sigma \to \mathbb{R}$ be a continuous function. Then the surface integral of f on Σ exists and is given by

$$\iint_D (f \circ \boldsymbol{r})(u,v) \big\| (\boldsymbol{r}_u \times \boldsymbol{r}_v)(u,v) \big\| \, d(u,v) \,.$$

Remark: If the surface Σ is the graph of a function $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$, then for a continuous function $g : \Sigma \to \mathbb{R}$, we have

$$\int_\Sigma g \, dS = \iint_R g\big(x, y, f(x,y)\big) \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2} \, dA \,.$$

§A.4 The Surface Integrals

Example

Evaluate the surface integral $\int_\Sigma (y^2 + 2yz)\,dS$, where Σ is the first-octant portion of the plane $2x + y + 2z = 6$.

First, we note that Σ can be parameterized by

$$\Sigma = \left\{ x\mathbf{i} + y\mathbf{j} + \frac{6-2x-y}{2}\mathbf{k} \,\middle|\, (x, y) \in R \right\},$$

where R is the triangle $\{(x, y) \mid x \in [0, 3], 0 \leqslant y \leqslant 6-2x\}$. Therefore,

$$\begin{aligned}
&\int_\Sigma (y^2 + 2yz)\,dS \\
&\quad = \iint_R \left(y^2 + 2y \cdot \frac{6-2x-y}{2}\right) \sqrt{1 + (-1)^2 + \left(-\frac{1}{2}\right)^2}\,dA \\
&\quad = \int_0^3 \left(\int_0^{6-2x} \frac{3}{2}(6y - 2xy)\,dy \right) dx = \cdots = \frac{243}{2}.
\end{aligned}$$

§A.4 The Surface Integrals

Example

Evaluate the surface integral $\displaystyle\int_{\Sigma} \sqrt{x(1+2z)}\, dS$, where Σ is the portion of the cylinder $z = \dfrac{y^2}{2}$ over the triangular region

$$R \equiv \left\{(x, y) \,\middle|\, x \geqslant 0, y \geqslant 0, x + y \leqslant 1\right\}$$

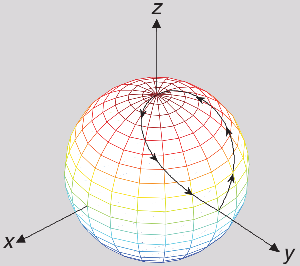
in the xy-plane. Similar to the previous example, we have

$$\begin{aligned}
\int_{\Sigma} \sqrt{x(1+2z)}\, dS &= \iint_R \sqrt{x(1+y^2)}\sqrt{1+0^2+y^2}\, dA \\
&= \int_0^1 \left(\int_0^{1-x} \sqrt{x}(1+y^2)\, dy\right) dx = \int_0^1 \sqrt{x}\left(y + \frac{y^3}{3}\right)\Big|_{y=0}^{y=1-x} dx \\
&= \int_0^1 \sqrt{x}\left(1 - x + \frac{(1-x)^3}{3}\right) dx = \frac{284}{945}\,.
\end{aligned}$$

§A.4 The Surface Integrals

Example

Evaluate the surface integral $\int_{\Sigma} z\,dS$, where Σ is the surface given in one of previous examples



which can be parameterized by

$$\Sigma = \Big\{ \boldsymbol{r}(\theta,\phi) = \cos\theta\sin\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\phi\,\mathbf{k} \,\Big|\, 0 \leqslant \phi \leqslant \frac{\pi}{2}, \phi \leqslant \theta \leqslant \pi - \phi \Big\}.$$

§A.4 The Surface Integrals

Example (cont.)

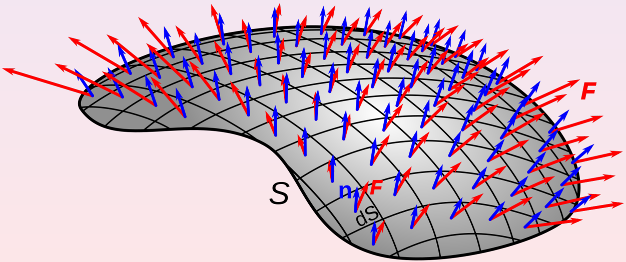
Therefore,

$$\begin{aligned}
\int_{\Sigma} z\,dS &= \int_0^{\frac{\pi}{2}} \Big(\int_{\phi}^{\pi-\phi} \cos\phi \big\| (\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi)(\theta,\phi) \big\|\, d\theta \Big)\, d\phi \\
&= \int_0^{\frac{\pi}{2}} \Big(\int_{\phi}^{\pi-\phi} \cos\phi \sin\phi\, d\theta \Big)\, d\phi \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\pi - 2\phi)\sin(2\phi)\, d\phi \\
&= \frac{1}{2} \Big[(\pi - 2\phi) \frac{-\cos(2\phi)}{2} \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\cos(2\phi)}{2}\, d\phi \Big] \\
&= \frac{1}{2} \Big(\frac{\pi}{2} - \frac{\sin(2\phi)}{4} \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \Big) = \frac{\pi}{4}\,.
\end{aligned}$$

§A.5 The Flux Integrals

Let $\Sigma \subseteq \mathbb{R}^3$ be a regular parametric surface with a continuous normal vector field $\mathbf{n} : \Sigma \to \mathbb{R}^3$ (sometimes this is called "Σ is oriented by $\mathbf{n}$"). For a bounded continuous vector-valued function $\boldsymbol{F} : \Sigma \to \mathbb{R}^3$, the flux integral of $\boldsymbol{F}$ across Σ (in direction $\mathbf{n}$) is the surface integral of $\boldsymbol{F} \cdot \mathbf{n}$ on Σ; that is,

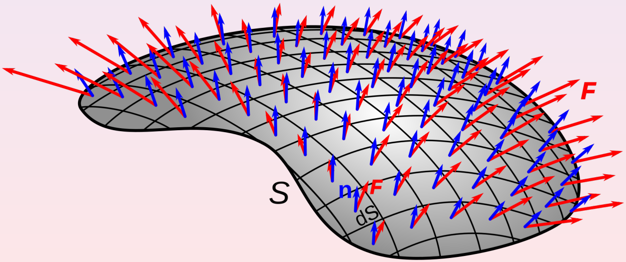
$$\text{the flux integral of } \boldsymbol{F} \text{ across } \Sigma \text{ (in direction } \mathbf{n}) = \int_\Sigma \boldsymbol{F} \cdot \mathbf{n}\, dS\,.$$



§A.5 The Flux Integrals

Let $\Sigma \subseteq \mathbb{R}^3$ be a regular parametric surface with a continuous normal vector field $\mathbf{n} : \Sigma \to \mathbb{R}^3$ (sometimes this is called "Σ is oriented by $\mathbf{n}$"). For a bounded continuous vector-valued function $\boldsymbol{F} : \Sigma \to \mathbb{R}^3$, the flux integral of $\boldsymbol{F}$ across Σ (in direction $\mathbf{n}$) is the surface integral of $\boldsymbol{F} \cdot \mathbf{n}$ on Σ; that is,

$$\text{the flux integral of } \boldsymbol{F} \text{ across } \Sigma \text{ (in direction } \mathbf{n}) = \int_{\Sigma} \boldsymbol{F} \cdot \mathbf{n} \, dS \, .$$



§A.5 The Flux Integrals

§A.5.1 Physical Interpretation

Let $\Omega \subseteq \mathbb{R}^3$ be an open set which stands for a fluid container and fully contains some liquid such as water, and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be a vector-field which stands for the fluid velocity; that is, $\boldsymbol{u}(x)$ is the fluid velocity at point $x \in \Omega$. Furthermore, let $\Sigma \subseteq \Omega$ be a surface immersed in the fluid with given orientation $\mathbf{n}$, and $c : \Omega \to \mathbb{R}$ be the concentration of certain material dissolving in the liquid. Then the amount of the material carried across the surface in the direction $\mathbf{n}$ by the fluid in a time period of Δt is

$$\Delta t \cdot \int_\Sigma c\boldsymbol{u} \cdot \mathbf{n}\, dS .$$

Therefore, $\int_\Sigma c\boldsymbol{u} \cdot \mathbf{n}\, dS$ is the rate of the amount of the material carried across the surface in the direction $\mathbf{n}$ by the fluid.

§A.5 The Flux Integrals

§A.5.1 Physical Interpretation

Let $\Omega \subseteq \mathbb{R}^3$ be an open set which stands for a fluid container and fully contains some liquid such as water, and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ be a vector-field which stands for the fluid velocity; that is, $\boldsymbol{u}(x)$ is the fluid velocity at point $x \in \Omega$. Furthermore, let $\Sigma \subseteq \Omega$ be a surface immersed in the fluid with given orientation $\mathbf{n}$, and $c : \Omega \to \mathbb{R}$ be the concentration of certain material dissolving in the liquid. Then the amount of the material carried across the surface in the direction $\mathbf{n}$ by the fluid in a time period of Δt is

$$\Delta t \cdot \int_\Sigma c\boldsymbol{u} \cdot \mathbf{n}\, dS .$$

Therefore, $\displaystyle\int_\Sigma c\boldsymbol{u} \cdot \mathbf{n}\, dS$ is the rate of the amount of the material carried across the surface in the direction $\mathbf{n}$ by the fluid.

§A.5 The Flux Integrals

Example

Find the flux integral of the vector field $\boldsymbol{F}(x, y, z) = (x, y^2, z)$ upward through the first octant part Σ of the cylindrical surface $x^2 + z^2 = a^2$, $0 < y < b$.

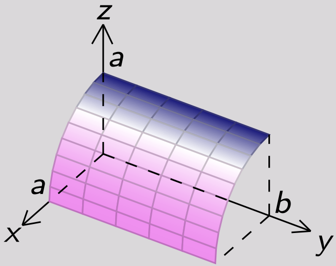


Figure 2: The surface Σ

§A.5 The Flux Integrals

Example (cont.)

First, we parameterize Σ by

$$\boldsymbol{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{a^2 - u^2}\mathbf{k}\,, \quad (u,v) \in D = (0,a) \times (0,b)$$

so that $\|(\boldsymbol{r}_u \times \boldsymbol{r}_v)(u,v)\|^2_{\mathbb{R}^3} = \dfrac{a^2}{a^2 - u^2}$, and the upward-pointing unit normal is $\mathbf{N}(x,y,z) = \left(\dfrac{x}{a}, 0, \dfrac{z}{a}\right)$. Therefore,

$$\begin{aligned}\int_\Sigma \boldsymbol{F} \cdot \mathrm{N}\, dS &= \iint_D \frac{1}{a}(u^2 + a^2 - u^2)\frac{a}{\sqrt{a^2 - u^2}}\, d(u,v) \\ &= a^2 \iint_D \frac{1}{\sqrt{a^2 - u^2}}\, d(u,v) \\ &= a^2 \int_0^b \int_0^a \frac{1}{\sqrt{a^2 - u^2}}\, du dv = a^2 b \arcsin \frac{u}{a}\Big|_{u=0}^{u=a} = \frac{\pi a^2 b}{2}\,.\end{aligned}$$

§A.5 The Flux Integrals

Example (cont.)

First, we parameterize Σ by

$$\boldsymbol{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \sqrt{a^2-u^2}\,\mathbf{k}\,, \quad (u,v) \in D = (0,a)\times(0,b)$$

so that $\|(\boldsymbol{r}_u \times \boldsymbol{r}_v)(u,v)\|^2_{\mathbb{R}^3} = \dfrac{a^2}{a^2-u^2}$, and the upward-pointing unit normal is $\mathbf{N}(x,y,z) = \left(\dfrac{x}{a}, 0, \dfrac{z}{a}\right)$. Therefore,

$$\begin{aligned}\int_\Sigma \boldsymbol{F}\cdot\mathbf{N}\,dS &= \iint_D \frac{1}{a}(u^2+a^2-u^2)\frac{a}{\sqrt{a^2-u^2}}\,d(u,v)\\ &= a^2\iint_D \frac{1}{\sqrt{a^2-u^2}}\,d(u,v)\\ &= a^2\int_0^b\int_0^a \frac{1}{\sqrt{a^2-u^2}}\,dudv = a^2 b\arcsin\frac{u}{a}\Big|_{u=0}^{u=a} = \frac{\pi a^2 b}{2}\,.\end{aligned}$$

§A.5 The Flux Integrals

§A.5.2 Measurements of the flux - the divergence operator

Let $\Omega \subseteq \mathbb{R}^3$ be an open set, and $\boldsymbol{u} = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field. Suppose that $\mathcal{O}$ is a bounded open set whose boundary is piecewise smooth so that an outward-pointing unit normal vector field $\mathbf{N} = (\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3)$ can be defined on $\partial\mathcal{O}$ except on some curves. Then the flux integral of $\boldsymbol{u}$ on $\partial\mathcal{O}$ in the direction $\mathbf{N}$ is

$$\int_{\partial\mathcal{O}} \boldsymbol{u} \cdot \mathrm{N}\, dS\,.$$

Consider a special case that $\mathcal{O} = (a_1, a_2) \times (b_1, b_2) \times (c_1, c_2)$ be an open cube so that $\partial\mathcal{O} = \{a_1, a_2\} \times \Sigma_1 \cup \{b_1, b_2\} \times \Sigma_2 \cup \{c_1, c_3\} \times \Sigma_3$. Then

$$\int_{\partial\mathcal{O}} \boldsymbol{u} \cdot \mathrm{N}\, dS = \sum_{k=1}^{3} \int_{\Sigma_k} \boldsymbol{u} \cdot \mathrm{N}\, dS\,.$$

§A.5 The Flux Integrals

§A.5.2 Measurements of the flux - the divergence operator

Let $\Omega \subseteq \mathbb{R}^3$ be an open set, and $\boldsymbol{u} = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field. Suppose that $\mathcal{O}$ is a bounded open set whose boundary is piecewise smooth so that an outward-pointing unit normal vector field $\mathbf{N} = (\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3)$ can be defined on $\partial\mathcal{O}$ except on some curves. Then the flux integral of $\boldsymbol{u}$ on $\partial\mathcal{O}$ in the direction $\mathbf{N}$ is

$$\int_{\partial\mathcal{O}} \boldsymbol{u} \cdot \mathrm{N}\, dS\,.$$

Consider a special case that $\mathcal{O} = (a_1, a_2) \times (b_1, b_2) \times (c_1, c_2)$ be an open cube so that $\partial\mathcal{O} = \{a_1, a_2\} \times \Sigma_1 \cup \{b_1, b_2\} \times \Sigma_2 \cup \{c_1, c_3\} \times \Sigma_3$. Then

$$\int_{\partial\mathcal{O}} \boldsymbol{u} \cdot \mathrm{N}\, dS = \sum_{k=1}^{3} \int_{\Sigma_k} \boldsymbol{u} \cdot \mathrm{N}\, dS\,.$$

§A.5 The Flux Integrals

Since on Σ_3 the outward-pointing normal $\mathbf{N}$ is given by

$$\mathbf{N}(x,y,z) = \begin{cases} -\mathbf{k} & \text{if } (x,y,z) \in [a_1,a_2] \times [b_1,b_2] \times \{c_1\}\,, \\ \mathbf{k} & \text{if } (x,y,z) \in [a_1,a_2] \times [b_1,b_2] \times \{c_2\}\,, \end{cases}$$

we find that

$$\begin{aligned} &\int_{\Sigma_3} \boldsymbol{u} \cdot \mathrm{N}\, dS \\ &\quad = \iint_{[a_1,a_2]\times[b_1,b_2]} u_3(x,y,c_2)\, dA - \iint_{[a_1,a_2]\times[b_1,b_2]} u_3(x,y,c_1)\, dA \\ &\quad = \iint_{[a_1,a_2]\times[b_1,b_2]} u_3(x,y,z)\Big|_{x=c_1}^{x=c_2} dA \\ &\quad = \iint_{[a_1,a_2]\times[b_1,b_2]} \Big(\int_{[c_1,c_2]} \frac{\partial u_3}{\partial z}(x,y,z)\, dz \Big)\, dA = \iiint_{\mathcal{O}} \frac{\partial u_3}{\partial z}\, dV, \end{aligned}$$

where the last equality is established by Fubini's Theorem.

§A.5 The Flux Integrals

Similarly,

$$\int_{\Sigma_1} \boldsymbol{u} \cdot \mathrm{N}\, dS = \iiint_{\mathcal{O}} \frac{\partial u_1}{\partial x}\, dV \quad \text{and} \quad \int_{\Sigma_2} \boldsymbol{u} \cdot \mathrm{N}\, dS = \iiint_{\mathcal{O}} \frac{\partial u_2}{\partial y}\, dV;$$

thus

$$\int_{\partial\mathcal{O}} \boldsymbol{u} \cdot \mathrm{N}\, dS = \iiint_{\mathcal{O}} \Big(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\Big)\, dV = \iiint_{\mathcal{O}} \mathrm{div}\boldsymbol{u}\, dV. \quad (1)$$

Remark: Let $\mathcal{O}(\boldsymbol{a}, r)$ denote a cube centered at $\boldsymbol{a} \in \Omega$ with side length r. Using (1),

$$\lim_{r\to 0} \frac{1}{|\mathcal{O}(\boldsymbol{a}, r)|} \int_{\partial\mathcal{O}(\boldsymbol{a},r)} \boldsymbol{u} \cdot \mathrm{N}\, dS = (\mathrm{div}\boldsymbol{u})(\boldsymbol{a}) \qquad \forall\, \boldsymbol{a} \in \Omega\,.$$

In other words, $\mathrm{div}\boldsymbol{u}$ at a point $\boldsymbol{x}$ is the instantaneous amount (per volume) of material (with concentration 1) carried outside an infinitesimal cube centered at $\boldsymbol{x}$.

§A.5 The Flux Integrals

Similarly,

$$\int_{\Sigma_1} \boldsymbol{u} \cdot \mathrm{N}\, dS = \iiint_{\mathcal{O}} \frac{\partial u_1}{\partial x}\, dV \quad \text{and} \quad \int_{\Sigma_2} \boldsymbol{u} \cdot \mathrm{N}\, dS = \iiint_{\mathcal{O}} \frac{\partial u_2}{\partial y}\, dV;$$

thus

$$\int_{\partial\mathcal{O}} \boldsymbol{u} \cdot \mathrm{N}\, dS = \iiint_{\mathcal{O}} \Big(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \Big)\, dV = \iiint_{\mathcal{O}} \mathrm{div}\, \boldsymbol{u}\, dV. \quad (1)$$

Remark: Let $\mathcal{O}(\boldsymbol{a}, r)$ denote a cube centered at $\boldsymbol{a} \in \Omega$ with side length r. Using (1),

$$\lim_{r \to 0} \frac{1}{|\mathcal{O}(\boldsymbol{a}, r)|} \int_{\partial\mathcal{O}(\boldsymbol{a}, r)} \boldsymbol{u} \cdot \mathrm{N}\, dS = (\mathrm{div}\, \boldsymbol{u})(\boldsymbol{a}) \qquad \forall\, \boldsymbol{a} \in \Omega\,.$$

In other words, $\mathrm{div}\,\boldsymbol{u}$ at a point $\boldsymbol{x}$ is the instantaneous amount (per volume) of material (with concentration 1) carried outside an infinitesimal cube centered at $\boldsymbol{x}$.

§A.6 The Divergence Theorem

Equation (1) from the previous page in fact holds for more general domain $\mathcal{O}$, and we have the following

Theorem (The Divergence Theorem)

Let $\Omega \subseteq R^3$ be a bounded domain such that $\partial\Omega$ is piecewise smooth with outward pointing normal $\mathbf{N}$, and $\boldsymbol{w} : \overline{\Omega} \to \mathbb{R}^3$ be continuously differentiable vector field. Then

$$\int_{\partial\Omega} \boldsymbol{w} \cdot \mathbf{N}\, dS = \iiint_{\Omega} \operatorname{div} \boldsymbol{w}\, dV .$$

Green's Theorem in Normal/Divergence Form: Let $\boldsymbol{F} : \bar{R} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable vector field. Then

$$\oint_{\partial R} \boldsymbol{F} \cdot \mathrm{N}\, ds = \iint_{R} \operatorname{div} \boldsymbol{F}\, dA ,$$

where N is the outward-pointing unit normal on ∂R.

§A.6 The Divergence Theorem

Equation (1) from the previous page in fact holds for more general domain $\mathcal{O}$, and we have the following

Theorem (The Divergence Theorem)

Let $\Omega \subseteq R^3$ be a bounded domain such that $\partial\Omega$ is piecewise smooth with outward pointing normal N, and $\boldsymbol{w} : \overline{\Omega} \to \mathbb{R}^3$ be continuously differentiable vector field. Then

$$\int_{\partial\Omega} \boldsymbol{w} \cdot \mathrm{N}\, dS = \iiint_{\Omega} \mathrm{div}\, \boldsymbol{w}\, dV .$$

Green's Theorem in Normal/Divergence Form: Let $\boldsymbol{F} : \bar{R} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable vector field. Then

$$\oint_{\partial R} \boldsymbol{F} \cdot \mathrm{N}\, ds = \iint_{R} \mathrm{div}\, \boldsymbol{F}\, dA ,$$

where N is the outward-pointing unit normal on ∂R.

§A.6 The Divergence Theorem

Remark: Similar to Green's Theorem in Divergence Form, the Divergence Theorem states that "一向量場在一區域的邊界上的某種有方向性的和（積分）等於該向量場某種微分的樣子（即散度）在該區域上的和（積分）":

$$\text{一向量場在一區域的邊界上的某種具方向性的和} = \int_{\partial\Omega} \boldsymbol{w}\cdot\mathbf{N}\,dS.$$

$$\text{該向量場某種微分的樣子在該區域上的和} = \iiint_{\Omega} \mathrm{div}\,\boldsymbol{w}\,dV.$$

Comparison: The fundamental theorem of calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a) \text{ “=” } \int_{\partial[a,b]} f.$$

§A.6 The Divergence Theorem

Remark: Similar to Green's Theorem in Divergence Form, the Divergence Theorem states that "一向量場在一區域的邊界上的某種有方向性的和（積分）等於該向量場某種微分的樣子（即散度）在該區域上的和（積分）":

$$\text{一向量場在一區域的邊界上的某種具方向性的和} = \int_{\partial\Omega} \boldsymbol{w}\cdot\mathbf{N}\,dS.$$

$$\text{該向量場某種微分的樣子在該區域上的和} = \iiint_{\Omega} \mathrm{div}\,\boldsymbol{w}\,dV.$$

Comparison: The fundamental theorem of calculus

$$\int_a^b f'(x)\,dx = f(b) - f(a) \text{ ``=''} \int_{\partial[a,b]} f.$$

§A.6 The Divergence Theorem

Remark: Similar to Green's Theorem in Divergence Form, the Divergence Theorem states that "一向量場在一區域的邊界上的某種有方向性的和（積分）等於該向量場某種微分的樣子（即散度）在該區域上的和（積分）":

$$\text{一向量場在一區域的邊界上的某種具方向性的和} = \int_{\partial\Omega} \boldsymbol{w}\cdot \mathbf{N}\, dS.$$

$$\text{該向量場某種微分的樣子在該區域上的和} = \iiint_{\Omega} \mathrm{div}\,\boldsymbol{w}\, dV.$$

Comparison: The fundamental theorem of calculus

$$\int_a^b f'(x)\, dx = f(b) - f(a) \text{ ``}=\text{''} \int_{\partial[a,b]} f.$$

§A.6 The Divergence Theorem

Letting $\boldsymbol{w}$ be the product of a scalar function φ and a vector field $\boldsymbol{v}$ in the Divergence Theorem, using the identity

$$\operatorname{div}(\varphi\boldsymbol{v}) = \varphi\operatorname{div}\boldsymbol{v} + \nabla\varphi\cdot\boldsymbol{v},$$

we conclude the following

Corollary

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain such that $\partial\Omega$ is piecewise smooth with outward-pointing unit normal $\mathbf{N}$, $\boldsymbol{v} : \overline{\Omega} \to \mathbb{R}^3$ be a continuously differentiable vector field, and $\varphi : \overline{\Omega} \to \mathbb{R}$ be continuously differentiable. Then

$$\iiint_{\Omega} \varphi\operatorname{div}\boldsymbol{v}\,dV = \int_{\partial\Omega} (\boldsymbol{v}\cdot\mathbf{N})\varphi\,dS - \iiint_{\Omega} \boldsymbol{v}\cdot\nabla\varphi\,dV.$$

§A.6 The Divergence Theorem

Letting $\boldsymbol{v} = f\mathbf{e}_i$ for some continuously differentiable function $f : \overline{\Omega} \to \mathbb{R}$ in the previous corollary, we obtain the following

Corollary

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain such that $\partial\Omega$ is piecewise smooth with outward-pointing normal $\mathbf{N} = (\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3)$, and $f, \varphi : \overline{\Omega} \to \mathbb{R}$ be continuously differentiable functions. Then

$$\iiint_\Omega \varphi \frac{\partial f}{\partial x_i}\, dV = \int_{\partial\Omega} f\varphi \mathrm{N}_i \, dS - \iiint_\Omega f \frac{\partial \varphi}{\partial x_i}\, dV .$$

§A.6 The Divergence Theorem

Example

Let Ω be the the first octant part bounded by the cylindrical surface $x^2 + z^2 = a^2$ and the plane $y = b$, and $\boldsymbol{F} : \Omega \to \mathbb{R}^3$ be a vector-valued function defined by $\boldsymbol{F}(x, y, z) = (x, y^2, z)$.

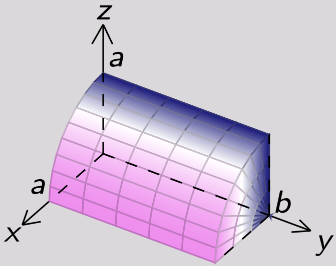


Figure 3: The domain Ω and its five pieces of boundaries

§A.6 The Divergence Theorem

Example (cont.)

With $\mathbf{N}$ denoting the outward-pointing unit normal of $\partial\Omega$,

$$\iiint_\Omega \operatorname{div} \boldsymbol{F}\, dV = \int_0^a \int_0^b \int_0^{\sqrt{a^2-x^2}} (2+2y)\, dzdydx$$
$$= (b^2+2b)\int_0^a \int_0^{\sqrt{a^2-x^2}} dzdx = \frac{\pi a^2(b^2+2b)}{4}.$$

On the other hand, we note that the boundary of Ω has five parts:

1. Σ as given in previous example,
2. two rectangles $\mathrm{R}_1 = \{x=0\}\times[0,b]\times[0,a]$, $\mathrm{R}_2 = [0,a]\times[0,b]\times\{z=0\}$, and
3. two quarter disc $\mathrm{D}_1 = \{(x,0,z)\in\mathbb{R}^3 \,|\, x^2+z^2 \leqslant a^2, x,z\geqslant 0\}$ and $\mathrm{D}_2 = \{(x,b,z)\in\mathbb{R}^3 \,|\, x^2+z^2 \leqslant a^2, x,z\geqslant 0\}$.

§A.6 The Divergence Theorem

Example (cont.)

With $\mathbf{N}$ denoting the outward-pointing unit normal of $\partial\Omega$,

$$\iiint_{\Omega} \operatorname{div} \boldsymbol{F}\, dV = \int_0^a \int_0^b \int_0^{\sqrt{a^2-x^2}} (2+2y)\, dzdydx$$

$$= (b^2+2b) \int_0^a \int_0^{\sqrt{a^2-x^2}} dzdx = \frac{\pi a^2(b^2+2b)}{4}\,.$$

On the other hand, we note that the boundary of Ω has five parts:

1. Σ as given in previous example,
2. two rectangles $\mathrm{R}_1 = \{x=0\} \times [0,b] \times [0,a]$, $\mathrm{R}_2 = [0,a] \times [0,b] \times \{z=0\}$, and
3. two quarter disc $\mathrm{D}_1 = \left\{(x,0,z) \in \mathbb{R}^3 \,\middle|\, x^2+z^2 \leqslant a^2, x, z \geqslant 0\right\}$ and $\mathrm{D}_2 = \left\{(x,b,z) \in \mathbb{R}^3 \,\middle|\, x^2+z^2 \leqslant a^2, x, z \geqslant 0\right\}$.

§A.6 The Divergence Theorem

Example (cont.)

Therefore,

$$\int_{\mathrm{R}_1} \boldsymbol{F} \cdot \mathrm{N}\, dS = \int_0^a \int_0^b (0, y^2, z) \cdot (-1, 0, 0)\, dydz = 0\,,$$

$$\int_{\mathrm{R}_2} \boldsymbol{F} \cdot \mathrm{N}\, dS = \int_0^a \int_0^b (x, y^2, 0) \cdot (0, 0, -1)\, dydx = 0\,,$$

$$\int_{\mathrm{D}_1} \boldsymbol{F} \cdot \mathrm{N}\, dS = \int_0^a \int_0^{\sqrt{a^2 - x^2}} (x, 0, z) \cdot (0, -1, 0)\, dzdx = 0\,,$$

and

$$\begin{aligned}\int_{\mathrm{D}_1} \boldsymbol{F} \cdot \mathrm{N}\, dS &= \int_0^a \int_0^{\sqrt{a^2 - x^2}} (x, b^2, z) \cdot (0, 1, 0)\, dzdx \\ &= b^2 \int_0^a \int_0^{\sqrt{a^2 - x^2}} dzdx = \frac{\pi a^2 b^2}{4}\,.\end{aligned}$$

§A.6 The Divergence Theorem

Example (cont.)

Together with the result in previous example, we find that

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{F}\cdot \mathrm{N}\, dS &= \Big(\int_{\Sigma} + \int_{\mathrm{R}_1} + \int_{\mathrm{R}_2} + \int_{\mathrm{D}_1} + \int_{\mathrm{D}_2}\Big) \boldsymbol{F}\cdot \mathrm{N}\, dS \\ &= \frac{\pi a^2 b^2}{4} + \frac{\pi a^2 b}{2} = \frac{\pi a^2 (b^2 + 2b)}{4} \\ &= \iiint_{\Omega} \mathrm{div}\boldsymbol{F}\, dV .\end{aligned}$$