Mathematical Modeling MA3067-* Midterm 2

National Central University, Dec. 6 2021

Problem 1. (20%) The speed v of a wave in deep water is determined by its wavelength λ and the acceleration g due to gravity. What does dimension analysis imply regarding the relationship between v, λ and g? Express v in terms of λ and g using Pi Theorem.

Solution. Choose fundamental dimension L (length) and T (time) so that $[v] = LT^{-1}$, $[\lambda] = L$ and $[g] = LT^{-2}$. Let $q_1 = v$, $q_2 = \lambda$ and $q_3 = g$. The dimension matrix D (in the order of dimension L, T) is

$$D = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \end{bmatrix}.$$

Clearly $\text{rank}(D) = 2$; thus by Pi Theorem there exist one dimensionless quanty $\pi = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3}$ such that the physical law is given by $\pi = k$ for some constant k. Such $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \alpha_3]^{\mathrm{T}}$ satisfies

$$D\boldsymbol{\alpha} = \mathbf{0} \qquad \text{or in the full form} \qquad \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A choice of such an α is $\big[1, -\frac{1}{2}, -\frac{1}{2}\big]^{\mathrm{T}}$ and the physical law is then equivalent to

$$\pi = q_1 q_2^{-1/2} q_3^{-1/2} = k$$

or $v = k\sqrt{\lambda g}$ for some constant k (to be determined by experiment data). □

Problem 2. Find a particular solution of $x''(t) - 3x'(t) - 4x(t) = 2\sin t$ through the following two methods.

1. (20%) Let $y = x' - 4x$. Show that $y'(t) + y(t) = 2\sin t$, and find a solution y then a solution x (to $x' - 4x = y$) using the method of integrating factor.

2. (20%) Make use of the formula

$$x_p(t) = -\varphi_1(t) \int \frac{g(t)\varphi_2(t)}{W[\varphi_1, \varphi_2](t)}\, dt + \varphi_2(t) \int \frac{g(t)\varphi_1(t)}{W[\varphi_1, \varphi_2](t)}\, dt$$

   for a particular solution x_p to the ODE

$$x''(t) + p(t)x'(t) + q(t)x(t) = g(t),$$

   where $\{\varphi_1, \varphi_2\}$ is a basis for the solution space $\big\{x : I \to \mathbb{R} \,\big|\, x''(t) + p(t)x'(t) + q(t)x(t) = 0\big\}$, and $W[\varphi_1, \varphi_2]$ is the Wronskian of φ_1 and φ_2.

In this problem you may need the integration formulas

$$\int e^{at} \sin(bt)\, dt = \frac{e^{at}}{a^2 + b^2}\big[a\sin(bt) - b\cos(bt)\big] + C,$$

$$\int e^{at} \cos(bt)\, dt = \frac{e^{at}}{a^2 + b^2}\big[b\sin(bt) + a\cos(bt)\big] + C.$$

Note that for a computation of a particular solution you can let $C = 0$ in the process of computations.

Solution. 1. If $y = x' - 4x$, then

$$y' + y = (x' - 4x)' + (x' - 4x) = x'' - 4x' + x' - 4x = x'' - 3x' - 4x$$

so that $y'(t) + y(t) = 2\sin t$. Therefore, the method of integrating factor shows that

$$\frac{d}{dt}\big[e^t y(t)\big] = 2e^t \sin t$$

which, with the help of the integration formula, implies that

$$e^t y(t) = e^t \sin t - e^t \cos t + C$$

or $y(t) = Ce^{-t} + \sin t - \cos t$. Let $C = 0$ and solve

$$x' - 4x = \sin t - \cos t\,.$$

By the method of integrating factor, we find that

$$\frac{d}{dt}\big[e^{-4t}x(t)\big] = e^{-4t}(\sin t - \cos t)\,;$$

thus the integration formula implies that

$$\begin{aligned} e^{-4t}x(t) &= \frac{1}{17}\Big[\big(-4e^{-4t}\sin t - e^{-4t}\cos t\big) - \big(e^{-4t}\sin t - 4e^{-4t}\cos t\big)\Big] + C \\ &= e^{-4t}\Big(-\frac{5}{17}\sin t + \frac{3}{17}\cos t\Big) + C\,. \end{aligned}$$

Therefore, a particular solution of the given ODE is given by

$$x_p(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t\,.$$

2. A basis $\{\varphi_1, \varphi_2\}$ for the solution space $\big\{x : I \to \mathbb{R} \,\big|\, x''(t) - 3x'(t) - 4x(t) = 0\big\}$ is $\varphi_1(t) = e^{-t}$ and $\varphi_2(t) = e^{4t}$ since the characteristic equation $r^2 - 3r - 4 = 0$ has two distinct zeros $r = 4$ and $r = -1$. Then

$$W[\varphi_1, \varphi_2](t) = \varphi_1(t)\varphi_2'(t) - \varphi_2(t)\varphi_1'(t) = e^{-t}\cdot 4e^{4t} - e^{4t}\cdot(-e^{-t}) = 5e^{3t}\,;$$

thus using the formula above we find that a particular solution to the given ODE is given by

$$x_p(t) = -e^{-t}\int \frac{2e^{4t}\sin t}{5e^{3t}}\,dt + e^{4t}\int \frac{2e^{-t}\sin t}{5e^{3t}}\,dt = -\frac{2}{5}e^{-t}\int e^t \sin t\,dt + \frac{2}{5}e^{4t}\int e^{-4t}\sin t\,dt\,.$$

Using the integration formula given in the problem we find that

$$\begin{aligned} x_p(t) &= -\frac{2}{5}e^{-t}\cdot\frac{1}{2}\big(e^t \sin t - e^t \cos t\big) + \frac{2}{5}e^{4t}\cdot\frac{1}{17}\big(-4e^{-4t}\sin t - e^{-4t}\cos t\big) \\ &= -\frac{1}{5}\big(\sin t - \cos t\big) - \frac{2}{5}\cdot\frac{1}{17}\big(4\sin t + \cos t\big) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t\,. \end{aligned}$$

□

Problem 3. (20%) Find a solution φ_2 of to the ODE $(1-t)x''(t) + tx'(t) - x(t) = 0$, $t \in [0, 1)$, so that $\{\varphi_1, \varphi_2\}$ spans the solution space of the ODE, where $\varphi_1(t) = e^t$.

Solution. Suppose that $\varphi_2(t) = v(t)\varphi_1(t) = v(t)e^t$ is a solution to the given ODE. Then

$$
\begin{aligned}
&(1-t)\big[v(t)e^t\big]'' + t\big[v(t)e^t\big]' - v(t)e^t = 0 \\
\Rightarrow\ &(1-t)\big[v''(t)e^t + 2v'(t)e^t + v(t)e^t\big] + t\big[v'(t)e^t + v(t)e^t\big] - v(t)e^t = 0 \\
\Rightarrow\ &(1-t)e^t v''(t) + (2-t)v'(t)e^t = 0 \\
\Rightarrow\ &(1-t)v''(t) + (2-t)v'(t) = 0\,.
\end{aligned}
$$

Let $y(t) = v'(t)$. Then y satisfies $y'(t) + \dfrac{2-t}{1-t}y(t) = 0$. Let $q(t) = \dfrac{2-t}{1-t} = 1 + \dfrac{1}{1-t}$. Then $Q(t) = t - \ln(1-t)$ is an integrating factor so that

$$\frac{d}{dt}\big[e^{Q(t)}y(t)\big] = 0\,.$$

Therefore, $y(t) = Ce^{-Q(t)} = Ce^{-t+\ln(1-t)} = Ce^{-t}(1-t)$, and (choosing $C = 1$) integrating by parts shows that

$$v(t) = \int y(t)\,dt = \int e^{-t}(1-t)\,dt = -e^{-t}(1-t) + \int e^{-t}(-1)\,dt = e^{-t}t\,.$$

This implies that $\varphi_2(t) = t$ is another solution to the given ODE. □

Problem 4. (20%) Find the solution to the initial value problem

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{f}(t)\,, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0\,,$$

where $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$, $\boldsymbol{f}(t) = \begin{bmatrix} -t^2 \\ -2t \\ -2 \\ e^t \\ 0 \end{bmatrix}$ and $\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$.

Solution. Rewrite the ODE as $\boldsymbol{x}'(t) - \boldsymbol{A}\boldsymbol{x}(t) = \boldsymbol{f}(t)$. Multiplying both sides by the integrating factor $e^{-t\boldsymbol{A}}$, we have

$$\frac{d}{dt}\big[e^{-t\boldsymbol{A}}\boldsymbol{x}(t)\big] = e^{-t\boldsymbol{A}}\boldsymbol{f}(t)\,.$$

Note that

$$e^{-tA}\boldsymbol{f}(t) = \begin{bmatrix} e^{-2t} & -te^{-2t} & \frac{t^2}{2}e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & -te^{-2t} & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & 0 & e^{-2t} & -te^{-2t} \\ 0 & 0 & 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -t^2 \\ -2t \\ -2 \\ e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2e^{-2t} \\ e^{-t} \\ 0 \end{bmatrix}.$$

Therefore,

$$\frac{d}{dt}\big[e^{-t\boldsymbol{A}}\boldsymbol{x}(t)\big] = \begin{bmatrix} 0 \\ 0 \\ -2e^{-2t} \\ e^{-t} \\ 0 \end{bmatrix}$$

so that

$$e^{-t\boldsymbol{A}}\boldsymbol{x}(t) - \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \int_0^t \begin{bmatrix} 0 \\ 0 \\ -2e^{-2s} \\ e^{-s} \\ 0 \end{bmatrix} ds = \begin{bmatrix} 0 \\ 0 \\ e^{-2t} - 1 \\ 1 - e^{-t} \\ 0 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{x}(t) = e^{t\boldsymbol{A}} \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \\ -e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 0 & e^{2t} & te^{2t} \\ 0 & 0 & 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \\ -e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{t^2}{2} \\ t \\ 1 \\ -e^{t} \\ 0 \end{bmatrix}. \qquad \square$$