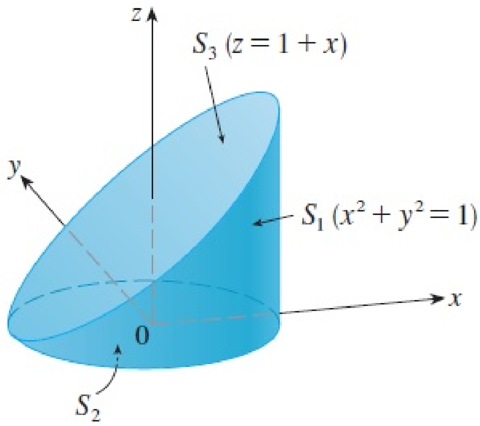
Mathematical Modeling MA3067-* Midterm 1

National Central University, Nov. 8 2021

Let S be the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2 \leqslant 1$ in the plane $z=0$, and whose top S_3 is the part of the plane $z=1+x$ that lies above S_2. See the following figure for reference.



Problem 1. (25%) Let C_1 be the curve enclosing S_3; that is, $C_1 = S_1 \cap S_3$. Find the line integral $\displaystyle\int_{C_1} \Big(\frac{x^2y^2}{\sqrt{1+y^2}} + ye^z\Big)\, ds$.

Solution. The curve C_1 can be parameterized by

$$\boldsymbol{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (1+\cos t)\mathbf{k}\,, \qquad t \in [0, 2\pi]\,.$$

Since

$$\|\boldsymbol{r}'(t)\| = \|-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} - \sin t\,\mathbf{k}\| = \sqrt{1+\sin^2 t}\,,$$

we have

$$\begin{aligned}\int_{C_1} \Big(\frac{x^2y^2}{\sqrt{1+y^2}} + ye^z\Big)\, ds &= \int_0^{2\pi} \Big(\frac{\cos^2 t \sin^2 t}{\sqrt{1+\sin^2 t}} + \sin t e^{1+\cos t}\Big)\sqrt{1+\sin^2 t}\, dt \\ &= \int_0^{2\pi} \sin^2 t \cos^2 t\, dt + \int_0^{2\pi} \sin t e^{1+\cos t}\sqrt{1+\sin^2 t}\, dt\,.\end{aligned}$$

For the first integral, using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we find that

$$\int_0^{2\pi} \sin^2 t \cos^2 t\, dt = \frac{1}{4}\int_0^{2\pi} \sin^2 2t\, dt = \frac{1}{4}\int_0^{2\pi} \frac{1-\cos 4t}{2}\, dt = \frac{1}{8}\Big(t - \frac{\sin 4t}{4}\Big)\Big|_{t=0}^{t=2\pi} = \frac{\pi}{4}\,.$$

For the second integral, we define $F(x) = \displaystyle\int e^{1+x}\sqrt{2-x^2}\, dx$; that is, F is an anti-derivative of the function $y = \sqrt{2-x^2}e^{1+x}$. Then the substitution of variable $u = \cos t$ implies that

$$\int \sin t e^{1+\cos t}\sqrt{1+\sin^2 t}\, dt = -\int e^{1+\cos t}\sqrt{2-\cos^2 t} d\cos t = -F(\cos t)\,;$$

thus

$$\int_0^{2\pi} \sin t e^{1+\cos t}\sqrt{1+\sin^2 t}\, dt = -F(\cos t)\Big|_{t=0}^{t=2\pi} = 0\,.$$

Therefore, $\displaystyle\int_{C_1} \Big(\frac{x^2y^2}{\sqrt{1+y^2}} + ye^z\Big)\, ds = \frac{\pi}{4}$. □

Problem 2. Let C_2 be the curve enclosing S_2 oriented counterclockwise. Evaluate the line integral $\displaystyle\oint_{C_2} y^3dx + xy^2dy$ by

1. (10%) computing the line integral directly;
2. (15%) applying Green's Theorem.

Solution. 1. First we note that C_2 (with counterclockwise orientation) can be parameterized by $\boldsymbol{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ with $t \in [0, 2\pi]$. Therefore,

$$\begin{aligned}\oint_{C_2} y^3dx + xy^2dy &= \int_0^{2\pi} \Big(\sin^3 t d(\cos t) + \cos t \sin^2 t d\sin t\Big) = \int_0^{2\pi} \big(\cos^2 t \sin^2 t - \sin^4 t\big) dt \\ &= \int_0^{2\pi} \sin^2 t \cos 2t\, dt = \int_0^{2\pi} \frac{1-\cos 2t}{2} \cos 2t\, dt = \frac{1}{2}\int_0^{2\pi} \big(\cos 2t - \cos^2 2t\big)\, dt \\ &= \frac{1}{2}\int_0^{2\pi} \Big(\cos 2t - \frac{1+\cos 4t}{2}\Big)\, dt = \frac{1}{2}\Big(\frac{\sin 2t}{2} - \frac{4t + \sin 4t}{8}\Big)\Big|_{t=0}^{t=2\pi} = -\frac{\pi}{2}\,.\end{aligned}$$

2. By Green's Theorem,

$$\oint_{C_2} y^3dx + xy^2dy = \int_{\{(x,y)|x^2+y^2\leqslant 1\}} \Big(\frac{\partial (xy^2)}{\partial x} - \frac{\partial y^3}{\partial y}\Big)\, dA = -2\int_{\{(x,y)|x^2+y^2\leqslant 1\}} y^2\, dA\,.$$

Using the polar coordinate, we obtain that

$$\begin{aligned}-2\int_{\{(x,y)|x^2+y^2\leqslant 1\}} y^2\, dA &= -2\int_0^{2\pi}\int_0^1 r^2 \sin^2\theta r dr d\theta = -2\int_0^{2\pi}\int_0^1 r^3 \sin^2\theta\, dr d\theta \\ &= -\frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta = -\frac{\pi}{2}\,.\end{aligned}$$

Therefore, $\displaystyle\oint_{C_2} y^3dx + xy^2dy = -\frac{\pi}{2}$. □

Problem 3. (25%) Find the surface integral $\displaystyle\int_{S_1} x^2y^2\, dS$.

Solution. The surface S_1 can be parameterized by

$$S_1 = \Big\{\boldsymbol{r}(u,v) \equiv \cos u\,\mathbf{i} + \sin u\,\mathbf{j} + v(1+\cos u)\mathbf{k} \,\Big|\, (u,v) \in [0,2\pi]\times[0,1]\Big\}\,.$$

Note that

$$\boldsymbol{r}_u(u,v) = -\sin u\,\mathbf{i} + \cos u\,\mathbf{j} - v\sin u\,\mathbf{k} \qquad \text{and} \qquad \boldsymbol{r}_v(u,v) = (1+\cos u)\mathbf{k}$$

so that

$$(\boldsymbol{r}_u \times \boldsymbol{r}_v)(u, v) = \cos u(1 + \cos u)\mathbf{i} + \sin u(1 + \cos u)\mathbf{j}\,.$$

Therefore,

$$\begin{aligned}
\int_{S_1} x^2y^2\,dS &= \int_0^{2\pi}\int_0^1 \cos^2 u \sin^2 u \big\| \cos u(1+\cos u)\mathbf{i} + \sin u(1+\cos u)\mathbf{j}\big\|\,dvdu \\
&= \int_0^{2\pi}\int_0^1 \cos^2 u\sin^2 u(1+\cos u)\,dvdu = \int_0^{2\pi} \cos^2 u \sin^2 u(1+\cos u)\,du \\
&= \frac{1}{4}\int_0^{2\pi} \sin^2 2u(1+\cos u)\,du = \frac{1}{4}\int_0^{2\pi} \frac{1-\cos 4u}{2}(1+\cos u)\,du \\
&= \frac{1}{8}\int_0^{2\pi} (1 - \cos 4u + \cos u - \cos 4u\cos u)\,du \\
&= \frac{1}{8}\Big[2\pi - \int_0^{2\pi} (\cos 5u + \cos 3u)\,du\Big] = \frac{\pi}{4}\,.
\end{aligned}$$

□

Problem 4. (25%) Let $\boldsymbol{F}(x, y, z) = (y\sin x + ye^z)\mathbf{i} - (x^2y + x\sin x + xe^z)\mathbf{j} + ye^z\mathbf{k}$. Verify the divergence theorem for the vector field $\boldsymbol{F}$ on the region enclosed by S.

Solution. On S_1 the outward-pointing unit normal is given by $\mathbf{N}(x, y, z) = x\mathbf{i} + y\mathbf{j}$. By the fact that

$$(\boldsymbol{F}\cdot\mathbf{N})(x, y, z) = xy(\sin x + e^z) - xy(xy + \sin x + e^z) = -x^2y^2 \qquad \forall\,(x, y, z) \in S_1\,,$$

Problem 3 shows that $\displaystyle\int_{S_1} \boldsymbol{F}\cdot\mathbf{N}\,dS = -\frac{\pi}{4}$.

On S_2 the outward-pointing unit normal is given by $\mathbf{N}(x, y, z) = -\mathbf{k}$. By the fact that

$$(\boldsymbol{F}\cdot\mathbf{N})(x, y, z) = -y \qquad \forall\,(x, y, z) \in S_2\,,$$

we find that

$$\int_{S_2} \boldsymbol{F}\cdot\mathbf{N}\,dS = -\int_{\{(x,y)|x^2+y^2\leqslant 1\}} y\,dA = \int_0^{2\pi}\int_0^1 r^2\sin\theta\,drd\theta = 0\,.$$

On S_3 the outward-pointing unit normal is given by $\mathbf{N}(x, y, z) = \dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$. By the fact that

$$(\boldsymbol{F}\cdot\mathbf{N})(x, y, z) = -\frac{y\sin x}{\sqrt{2}} \qquad \forall\,(x, y, z) \in S_3\,,$$

we find that

$$\begin{aligned}
\int_{S_3} \boldsymbol{F}\cdot\mathbf{N}\,dS &= -\int_{\{(x,y)|x^2+y^2\leqslant 1\}} \frac{y\sin x}{\sqrt{2}}\sqrt{1^2+0^2}\,dA = -\frac{1}{\sqrt{2}}\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\sin x\,dydx \\
&= -\sqrt{2}\int_{-1}^1 \sqrt{1-x^2}\sin x\,dx = 0\,,
\end{aligned}$$

where we use the fact that the function $y = \sqrt{1-x^2}\sin x$ is an odd function to conclude the last equality. Therefore,

$$\int_{\partial\Omega} \boldsymbol{F}\cdot\mathbf{N}\,dS = \sum_{i=1}^{3}\int_{S_i} \boldsymbol{F}\cdot\mathbf{N}\,dS = -\frac{\pi}{4}\,.$$

Finally, since $(\mathrm{div}\boldsymbol{F})(x,y,z) = y\cos x - x^2 + ye^z$, we find that

$$\begin{aligned}\int_\Omega \mathrm{div}\boldsymbol{F}\,dV &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1+x}\big(y\cos x - x^2 + ye^z\big)\,dzdydx\\ &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\big[(1+x)(y\cos x - x^2) + y(e^{1+x}-1)\big]\,dydx\\ &= -2\int_{-1}^{1}(1+x)x^2\sqrt{1-x^2}\,dx\,.\end{aligned}$$

Making the substitute of variable $x = \sin u$ (so $dx = \cos u\,du$), we find that

$$\begin{aligned}\int_\Omega \mathrm{div}\boldsymbol{F}\,dV &= -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin u)\sin^2 u\cos^2 u\,du = -2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big[\frac{\sin^2 2u}{4} + \sin u(1-\cos^2 u)\cos^2 u\Big]\,du\\ &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos 4u}{4}\,du + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2 u - \cos^4 u)d\cos u\\ &= \Big(\frac{2}{3}\cos^3 u - \frac{2}{5}\cos^5 u - \frac{u}{4} + \frac{\sin 4u}{16}\Big)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{\pi}{4}\,.\end{aligned}$$

Therefore, we conclude that $\displaystyle\int_\Omega \mathrm{div}\boldsymbol{F}\,dV = \int_{\partial\Omega}\boldsymbol{F}\cdot\mathbf{N}\,dS$. □