7. Similitude and Dimensional Analysis
(相似性與因次分析)

●**真實流體流況大多非常複雜，僅能以解析法**(Analytical method)**求得近似解**(Approximate solution)

⇒ **進行實驗模擬真實流況有助解決問題**

●**實驗**(Experiments)**分成兩種：**

①**物理模型試驗**(Physical modeling)

又稱水工模型試驗

②**數值模式試驗**(Numerical modeling)

又稱數值模擬(Numerical simulation)

●**進行物理或數值實驗常遭遇之困難：**

①**變數(變因)太多，不易控制**

②**真實流況之尺度常太大或太小，無法以真實尺寸進行實驗**

③**數值模擬需要靠水工模型試驗數據或實地量測數據進行模式之參數檢定**(Calibration)**與結果驗証**(Verification)。

●**進行水工模型試驗之前要先分析：**

①**物理模型須遵守哪種<u>相似律</u>**(Law of Similitude)

②**哪些是重要之變數，由<u>因次分析</u>**(Dimensional analysis)**得之**

●**常用之物理量因次(常採**MLT**或**FLT)

物理量	符號	因次
(1)質量	M	M (or $F = Ma \Rightarrow M = \frac{F}{a} = \frac{F}{\left(\frac{L}{T^2}\right)} = \frac{FT^2}{L}$)
(2)長度	L	L
(3)時間	T	T
(4)速度	V	L/T
(5)加速度	a	L/T^2
(6)力	F	F (or $F = Ma = \frac{ML}{T^2}$)

(7)**應力(或壓力)** t (or p) $\frac{F}{A}=\frac{F}{L^2}$ (or $\frac{Ma}{A}=\frac{ML}{T^2L^2}=\frac{M}{T^2L}$)

(8)**密度** r $r=\frac{M}{V}=\frac{M}{L^3}$ (or $\frac{F}{aV}=\frac{F}{L/T^2\cdot L^3}=\frac{FT^2}{L^4}$)

(9)**比重量** g $g=r\,g=\left(\frac{M}{L^3}\right)\left(\frac{L}{T^2}\right)=\frac{M}{L^2T^2}$ (or $\frac{F}{L^3}$)

(10)**動力黏滯性** m $\frac{F}{A}=m\frac{du}{dy}\Rightarrow m=\frac{F}{A}\cdot\frac{y}{u}=\frac{\left(M\cdot\frac{L}{T^2}\right)}{L^2}\cdot\frac{L}{\left(\frac{L}{T}\right)}=\frac{M}{TL}$

(11)**運動黏滯性** n $n=\frac{m}{r}=\frac{(M/TL)}{(M/L^3)}=\frac{L^2}{T}$

(12)**流量** Q $Q=AV=(L^2)(L/T)=L^3/T$

(13)**功或能** W (or E) $W=F\cdot L=\frac{ML^2}{T^2}$

(14)**功率** P $P=\frac{W}{T}=FV=ML^2/T^3$

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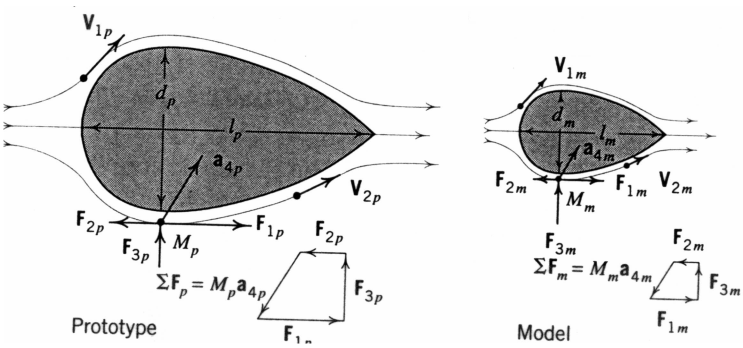
1. Similitude and Physical Model (物理模型相似性)

●**<u>模型相似</u>有下列三種：**



①**幾何相似**(Geometric Similarity)**：指模型**(Model)**與原型**(Prototype)**間之幾何尺寸成比例**。

$$\Rightarrow\ \frac{d_p}{d_m}=\frac{l_p}{l_m},\quad \frac{A_p}{A_m}=\left(\frac{d_p}{d_m}\right)^2=\left(\frac{l_p}{l_m}\right)^2\ \text{etc.}$$

②**運動相似**(Kinematic similarity)：**指模型與原型間相對應點之速度與加速度成比例。**

$$\frac{V_{1p}}{V_{1m}} = \frac{V_{2p}}{V_{2m}}\text{，}\quad \frac{a_{3p}}{a_{3m}} = \frac{a_{4p}}{a_{4m}}$$

③**動力相似**(Dynamic Similarity)：**指模型與原型間相對應點所受之各種外力成比例。外力可為分力或合力(各種分力包括：**gravity，viscous，pressure....)

$$\frac{F_{3p}}{F_{3m}} = \frac{F_{4p}}{F_{4m}} = \frac{M_p a_{3p}}{M_m a_{3m}} = \frac{M_p a_{4p}}{M_m a_{4m}}$$

⇒ **完整之模型相似，須同時符合三者：①幾何 ②運動 ③動力 相似**

●**流場中常考慮之力：**(6**種**)

①**慣性力**(Inertia)， F_I

$$F_I = Ma = \left(r L^3\right)\left(\frac{V^2}{L}\right) = rV^2 L^2 \qquad \left(= r AV^2 = r QV\right)$$

②**壓力**(Pressure)， F_p

$$F_p = PA = P \cdot L^2$$

③**重力**(Gravity)， F_G

$$F_G = Mg = r g L^3$$

④**黏滯力**(Viscosity)， F_V

$$F_V = m\left(\frac{du}{dy}\right)A = m\left(\frac{V}{L}\right)L^2 = mVL$$

⑤**表面張力**(Surface Tension)， F_T

$$F_T = s L$$

⑥**彈性力**(Elasticity)， F_E

$$F_E = EA = EL^2$$

●**流體力學常用之無因次參數**(Dimensionless Numbers)

①**雷諾數**(Reynolds number)， $\mathrm{Re} = \frac{VL}{n}$

$$\mathrm{Re} = \frac{\text{Inertia force}}{\text{Viscosity force}} = \frac{rV^2L^2}{mVL} = \frac{rVL}{m} = \frac{VL}{n}$$

②**福祿數**(Froude number)**，** $\mathrm{Fr}=\dfrac{V}{\sqrt{gL}}$

$$\mathrm{Fr}^2=\frac{\text{Inertia force}}{\text{Gravity force}}=\frac{rV^2L^2}{r\,gL^3}=\frac{V^2}{gL}$$

③**尤拉數**(Euler number)**，** $\mathrm{Eu}=V\sqrt{\dfrac{r}{2\cdot\Delta p}}$

$$\mathrm{Eu}^2=\frac{\frac{1}{2}\text{Inertia force}}{\text{Pressure force}}=\frac{\frac{1}{2}rV^2L^2}{\Delta p\cdot L^2}=\frac{V^2r}{2\cdot\Delta p}$$

④**韋伯數**(Weber number)**，** $\mathrm{W}=\dfrac{rV^2L}{s}$

$$\mathrm{W}=\frac{\text{Inertia force}}{\text{Surface tension force}}=\frac{rV^2L^2}{s\,L}=\frac{rV^2L}{s}$$

⑤**馬赫數**(Mach number)**，** $\mathrm{M}=V\sqrt{\dfrac{r}{E}}$

$$\mathrm{M}^2=\frac{\text{Inertia force}}{\text{Elasticity force}}=\frac{rV^2L^2}{EL^2}=\frac{rV^2}{E}$$

⇒**若要達到"動力完全相似"，必須模型與原型之所有無因次參數要相等，即** $(\mathrm{Re})_p=(\mathrm{Re})_m$ ， $(\mathrm{Fr})_p=(\mathrm{Fr})_m$ ， $(\mathrm{Eu})_p=(\mathrm{Eu})_m$ ， $(\mathrm{W})_p=(\mathrm{W})_m$ ， $(\mathrm{M})_p=(\mathrm{M})_m\ldots$

但模型試驗要符合每一種相似律在實際上十分困難，且幾乎不可能！所以通常只考慮流場中<u>最重要之特性</u>，而要求其相似。

⇒**故最重要者為：**

①**黏滯力** → Reynolds **相似律，** $(\mathrm{Re})_p=(\mathrm{Re})_m$

②**重力** → Froude **相似律，** $(\mathrm{Fr})_p=(\mathrm{Fr})_m$

③**壓力** → Euler **相似律，** $(\mathrm{Eu})_p=(\mathrm{Eu})_m$

④**表面張力** → Weber **相似律，** $(\mathrm{W})_p=(\mathrm{W})_m$

◎**一般而言**①**管流(黏滯力主控)** → **遵守**Reynolds **相似律**

②**明渠流(重力主控)** → **遵守**Froude **相似律**

2. Dimensional Analysis (因次分析)

●Fourier's <u>Principle of dimensional homogeneity</u> **(因次齊次定理)**

"The dimensions of each side of the equation must be the same."

例： Suppose we know $P = f(Q, g, E_T)$

where P = power of hydraulic turbine

Q = flowrate through machine

g = specific weight of fluid

E_T = unit mechanical energy

$$\Rightarrow P = k \cdot Q^a g^b E_T^c$$

Dimensions：

$$\frac{ML^2}{T^3} = \left(\frac{L^3}{T}\right)^a \left(\frac{M}{L^2T^2}\right)^b (L)^c$$

	L.H.S.		R.H.S.		
M：	1	=	b		a=1
L：	2	=	3a-2b+c	⇒	b=1
T：	-3	=	-a-2b		c=1

$$\therefore P = kQg E_T$$

此種分析方法有缺點：只能有三個變數(Q, g, E_T)，**因為只有三個獨立物理量**(M, L, T)

若 $A = k \cdot B^a \cdot C^b \cdot D^c \cdot E^d$ ⇒**則無法求出**a, b, c, d

● Buckingham p -theorem (**白金漢** p **定理**)

①**列出所有相關之物理量** $q_1, q_2, q_3, \ldots\ldots, q_n$

②**其中** q_1 **為因變數，** $q_2, q_3, \ldots\ldots, q_n$ **為變數**

i.e. $q_1 = f(q_2, q_3, q_4, \ldots\ldots, q_n)$

③**這些物理量與**k**個物理因次有關**

(e.g. **如果與**L.M.T**有關** k=3

如果只與L.M**有關** k=2，**與**M.T**有關**k=2

如果只與L**有關** k=1，**與**M**有關**k=1…….)

④**這些物理量** $q_1, q_2, q_3, \ldots\ldots, q_n$ **可組成**(n-k)**個無因次組合** $p_1, p_2, p_3, \ldots\ldots, p_{n-k}$

⑤**這些無因次組合之關係如下：**

$$p_1 = f(p_2, p_3, \ldots\ldots\ldots, p_{n-k})$$

3. Nondimensionalization of Equations (方程式之無因次化)

- Navier-Stokes eqn. (x -direction)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g_x-\frac{1}{r}\frac{\partial p}{\partial x}+n\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Select $L\equiv$ Characteristic length

$V\equiv$ Charcteristic velocity

Define $x_0=x/L$, $y_0=y/L$, $z_0=z/L$

$u_0=u/V$, $v_0=v/V$, $w_0=w/V$

$t_0=t/(L/V)$, $p_0=p/(rV^2)$

$\Rightarrow$ $x=x_0L$, $y=y_0L$, $z=z_0L$

$u=u_0V$, $v=v_0V$, $w=w_0V$

$t=t_0(L/V)$, $p=p_0(rV^2)$

代入N-S eqn.

$$\left(\frac{V^2}{L}\right)\left(\frac{\partial u_0}{\partial t_0}+u_0\frac{\partial u_0}{\partial x_0}+v_0\frac{\partial u_0}{\partial y_0}+w_0\frac{\partial u_0}{\partial z_0}\right)$$

$$=-g_x-\left(\frac{V^2}{L}\right)\frac{\partial p_0}{\partial x_0}+\left(\frac{nV}{L^2}\right)\left(\frac{\partial^2 u_0}{\partial x_0{}^2}+\frac{\partial^2 u_0}{\partial y_0{}^2}+\frac{\partial^2 u_0}{\partial z_0{}^2}\right)$$

$$\Rightarrow\frac{\partial u_0}{\partial t_0}+u_0\frac{\partial u_0}{\partial x_0}+v_0\frac{\partial u_0}{\partial y_0}+w_0\frac{\partial u_0}{\partial z_0}$$

$$=-\left(\frac{g_xL}{V^2}\right)-\frac{\partial p_0}{\partial x_0}+\left(\frac{n}{VL}\right)\left(\frac{\partial^2 u_0}{\partial x_0{}^2}+\frac{\partial^2 u_0}{\partial y_0{}^2}+\frac{\partial^2 u_0}{\partial z_0{}^2}\right)$$

$$=-\left(\frac{1}{\mathrm{Fr}^2}\right)-\frac{\partial p_0}{\partial x_0}+\left(\frac{1}{\mathrm{Re}}\right)\left(\frac{\partial^2 u_0}{\partial x_0{}^2}+\frac{\partial^2 u_0}{\partial y_0{}^2}+\frac{\partial^2 u_0}{\partial z_0{}^2}\right)$$

- Given **Fr** and **Re** $\Rightarrow$ Solution is valid for all scales of flows.

- Solution is irrelevant to the unit system.

7. Similitude and Dimensional Analysis
(相似性 與 因次分析)

- 真實流體流況大多非常複雜，解析法 (Analytical method) 僅能利用簡化之假設條件求得 近似解 (Approximate solution)

  ➩ 進行實驗 模擬 真實流況，有助於 瞭解問題 及 解決問題

- 實驗 (Experiments) 可分成兩種：

  ① 物理模型試驗 (Physical experiment, or physical modeling)

  又稱 水工模型試驗 (Scale model experiment)

  ② 數值模型試驗 (Numerical experiment, or numerical modeling)

  又稱 數值模擬 (Numerical simulation)

1

- 進行 物理模型試驗 或 數值模型試驗 常遭遇之問題：

  ①真實流況之尺度 常 太大 或 太小，無法以 真實尺寸 進行實驗

  ②實驗變數(變因) 太多，不易控制 (全範圍涵蓋 過於費時)

  ③ 電腦數值模擬 雖較方便，但需靠 水工模型試驗 或 現地實測數據 進行模式之 參數檢定 (Calibration) 與 結果驗証 (Verification)。

- 進行 水工模型試驗 之前 要先分析：

  ① 物理模型須遵守哪種 相似律 (Law of Similitude)

  ② 由 因次分析 (Dimensional analysis) 得到 重要變數間之關係，有利於 實驗設計 與 結果分析

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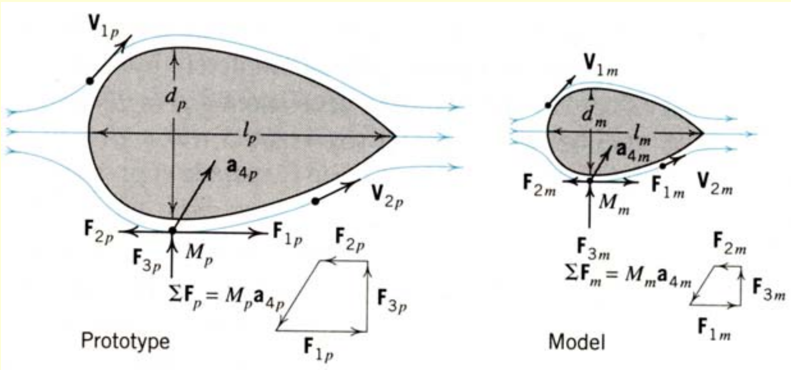
● 常用 物理量之 因次 (採 MLT 或 FLT 制)

物理量	符號	因次
(1) 質量	m	M (or $F = ma \Rightarrow m = \frac{F}{a} = \frac{F}{\left(\frac{L}{T^2}\right)} = \frac{FT^2}{L}$)
(2) 長度	L	L
(3) 時間	t	T
(4) 速度	U	L/T
(5) 加速度	a	L/T^2
(6) 力	F	F (or $F = ma = \frac{ML}{T^2}$)
(7) 應力(或壓力)	τ (or p)	$\frac{F}{A} = \frac{F}{L^2}$ (or $\frac{ma}{A} = \frac{ML}{T^2L^2} = \frac{M}{T^2L}$)

(8) 密度	ρ	$\rho = \frac{m}{V} = \frac{M}{L^3}$ (or $\frac{F}{aV} = \frac{F}{L/T^2 \cdot L^3} = \frac{FT^2}{L^4}$)
(9) 比重量	γ	$\gamma = \rho g = \left(\frac{M}{L^3}\right)\left(\frac{L}{T^2}\right) = \frac{M}{L^2T^2}$ (or $\frac{F}{L^3}$)
(10)動力黏滯性	μ	$\frac{F}{A} = \mu\frac{du}{dy} \Rightarrow \mu = \frac{F}{A}\cdot\frac{y}{u} = \frac{\left(M\cdot\frac{L}{T^2}\right)}{L^2}\cdot\frac{L}{\left(\frac{L}{T}\right)} = \frac{M}{TL}$
(11)運動黏滯性	ν	$\nu = \frac{\mu}{\rho} = \frac{(M/TL)}{(M/L^3)} = \frac{L^2}{T}$
(12) 流量	Q	$Q = AV = (L^2)(L/T) = L^3/T$
(13) 功或能	W (or E)	$W = F \cdot L = \frac{ML^2}{T^2}$
(14) 功率	P	$P = \frac{W}{t} = FU = ML^2/T^3$

1. Similitude and Physical Model (物理模型相似性)

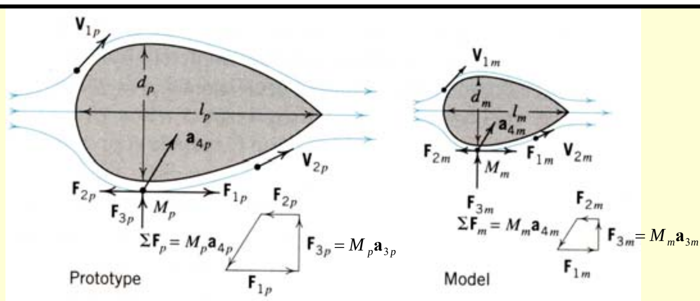
- 模型相似 包括 下列三種：



① 幾何相似 (**Geometric Similarity**)：模型 (**Model**) 與 原型 (**Prototype**) 之幾何尺寸成比例

$$\Rightarrow \frac{d_p}{d_m}=\frac{l_p}{l_m} \;,\; \frac{A_p}{A_m}=\left(\frac{d_p}{d_m}\right)^2=\left(\frac{l_p}{l_m}\right)^2 \text{ etc.}$$

5



② 運動相似 (**Kinematic similarity**)：模型 與 原型 相對應點之 速度與 加速度 成比例

$$\frac{V_{1p}}{V_{1m}}=\frac{V_{2p}}{V_{2m}} \;,\; \frac{a_{3p}}{a_{3m}}=\frac{a_{4p}}{a_{4m}}$$

③ 動力相似 (**Dynamic Similarity**)：

模型 與 原型 相對應點 所受之各種外力 成比例，外力包括 合力 與 分力 (各種分力包括：**gravity**，**viscous**，**pressure ...**)

$$\frac{F_{3p}}{F_{3m}}=\frac{F_{4p}}{F_{4m}}$$

➪ 完全相似 (**Complete similarity**) 須同時符合 ①幾何 ②運動 ③動力 相似

● 流場中 常考慮之力：(6 種)

① 慣性力 (Inertial Force)， F_I

$$F_I = ma = (\rho L^3)\left(\frac{V^2}{L}\right) = \rho V^2 L^2 \qquad (= \rho A V^2 = \rho Q V)$$

② 壓力 (Pressure Force)， F_p

$$F_p = PA = P \cdot L^2$$

③ 重力 (Gravity Force)， F_G

$$F_G = mg = \rho g L^3$$

④ 黏滯力 (Viscous Force)， F_V

$$F_V = \mu\left(\frac{du}{dy}\right)A = \mu\left(\frac{V}{L}\right)L^2 = \mu V L$$

⑤ 表面張力 (Surface Tension Force)， F_T

$$F_T = \sigma L$$

⑥ 彈性力 (Elastic Force)， F_E

$$F_E = EA = EL^2$$

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● 流體力學 常用之 無因次參數 (**Dimensionless Numbers**)

① 雷諾數 (Reynolds number)，$\mathrm{Re} = \frac{VL}{\nu}$

$$\mathrm{Re} = \frac{\text{Inertia force}}{\text{Viscosity force}} = \frac{\rho V^2 L^2}{\mu V L} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$$

② 福祿數 (Froude number)，$\mathrm{Fr} = \frac{V}{\sqrt{gL}}$

$$\mathrm{Fr}^2 = \frac{\text{Inertia force}}{\text{Gravity force}} = \frac{\rho V^2 L^2}{\rho g L^3} = \frac{V^2}{gL}$$

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③ 尤拉數 (Euler number)，$\mathbf{Eu} = V\sqrt{\dfrac{\rho}{2\cdot\Delta p}}$

$$\mathbf{Eu}^2 = \frac{\frac{1}{2}\text{Inertia force}}{\text{Pressure force}} = \frac{\frac{1}{2}\rho V^2 L^2}{\Delta p \cdot L^2} = \frac{V^2\rho}{2\cdot\Delta p}$$

④ 韋伯數 (Weber number)，$\mathbf{W} = \dfrac{\rho V^2 L}{\sigma}$

$$\mathbf{W} = \frac{\text{Inertia force}}{\text{Surface tension force}} = \frac{\rho V^2 L^2}{\sigma L} = \frac{\rho V^2 L}{\sigma}$$

⑤ 馬赫數 (Mach number)，$\mathbf{M} = V\sqrt{\dfrac{\rho}{E}} \left(= \dfrac{V}{c} \right)$

$$\mathbf{M}^2 = \frac{\text{Inertia force}}{\text{Elasticity force}} = \frac{\rho V^2 L^2}{E L^2} = \frac{\rho V^2}{E}$$

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➪ “完全動力相似” 必須 模型 與 原型 之 所有無因次參數 要相等

$$(\mathrm{Re})_p = (\mathrm{Re})_m,\ (\mathrm{Fr})_p = (\mathrm{Fr})_m,\ (\mathrm{Eu})_p = (\mathrm{Eu})_m,\ (\mathrm{W})_p = (\mathrm{W})_m,\ (\mathrm{M})_p = (\mathrm{M})_m \ \ldots$$

但 模型試驗 要符合 完全動力相似 實際上 十分困難，且 幾乎不可能！

例如： Length ratio $\dfrac{L_p}{L_m} = 4$, viscosity ratio $\dfrac{\nu_p}{\nu_m} = 1$, gravity ratio $\dfrac{g_p}{g_m} = 1$

(1) **Reynolds similarity :** $(\mathrm{Re})_p = (\mathrm{Re})_m \Rightarrow \dfrac{V_p L_p}{\nu_p} = \dfrac{V_m L_m}{\nu_m}$

$\Rightarrow$ Velocity ratio $\dfrac{V_p}{V_m} = \dfrac{L_m}{L_p}\dfrac{\nu_p}{\nu_m} = \left(\dfrac{1}{4}\right)(1) = \boxed{\dfrac{1}{4}}$

(2) **Froude similarity :** $(\mathrm{Fr})_p = (\mathrm{Fr})_m \Rightarrow \dfrac{V_p}{\sqrt{g_p L_p}} = \dfrac{V_m}{\sqrt{g_m L_m}}$

$\Rightarrow$ Velocity ratio $\dfrac{V_p}{V_m} = \sqrt{\dfrac{g_p}{g_m}}\sqrt{\dfrac{L_p}{L_m}} = (1)\sqrt{4} = \boxed{2}$

$\Rightarrow$ **Reynolds and Froude similarity are not met simultaneously**!

因此，通常 只考慮 流場中 最重要之特性，而要求其相似。

➪ 故最重要者為：

① 黏滯力 → **Reynolds** 相似律， $(\mathrm{Re})_p = (\mathrm{Re})_m$

② 重力 → **Froude** 相似律， $(\mathrm{Fr})_p = (\mathrm{Fr})_m$

③ 壓力 → **Euler** 相似律， $(\mathrm{Eu})_p = (\mathrm{Eu})_m$

④ 表面張力 → **Weber** 相似律， $(\mathrm{W})_p = (\mathrm{W})_m$

◎ 一般而言：管流 或 邊界層 (黏滯力主控) → 遵守 **Reynolds** 相似律

明渠流 (重力主控) → 遵守 **Froude** 相似律

(Rule of thumb !!)

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2. Dimensional Analysis (因次分析)

● **Fourier's Principle of dimensional homogeneity** (因次齊次定理)

"The dimensions on each side of an equation must be the same."

例如：**Suppose we know** $P = f(Q, \gamma, E_T)$

where P = **power of hydraulic turbine**

Q = **flowrate through machine**

γ = **specific weight of fluid**

E_T = **unit mechanical energy**

Fourier (French, 1768–1830)

$$\Rightarrow P = Q^a \gamma^b E_T^c$$

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Dimensions： $\frac{ML^2}{T^3} = \left(\frac{L^3}{T}\right)^a \left(\frac{M}{L^2T^2}\right)^b (L)^c$

	L.H.S.		R.H.S.		
M :	1	=	b		a=1
L :	2	=	3a-2b+c	⇨	b=1
T :	-3	=	-a-2b		c=1

$$\therefore \quad P = Q\gamma E_T$$

此種分析方法有缺點：最多只能有三個變數 (Q，γ，E_T)，因為只有三個獨立因次(M，L，T)用來求解 a, b, c

若 $A = B^a \cdot C^b \cdot D^c \cdot E^d$

⇨ 則無法求出 a，b，c，d

- **Buckingham π theorem (白金漢 π 定理)**

① 列出所有相關之物理量 q_1，q_2，q_3 ,....., q_n

② 其中 q_1 為因變數，q_2，q_3 ,....., q_n 為自變數

i.e. $q_1 = f(q_2, q_3, q_4, ..., q_n)$

③ 這些物理量 與 k個 物理因次 有關

(e.g. 如果與 LMT有關 k=3
如果只與 LM有關 k=2，或只與MT有關 k=2
如果只與 L有關 k=1，或只與M有關 k=1)

④ 這些物理量 q_1，q_2，q_3 ,....., q_n 可組成 (n-k)個 無因次參數組合

π_1，π_2，π_3 ,....., π_{n-k}

⑤ 這些 無因次參數組合 之關係，可表示如下：

$\pi_1 = f(\pi_2, \pi_3, ..., \pi_{n-k})$

Buckingham (American, 1914)

3. Non-dimensionalize Equations (方程式之無因次化)

- Navier-Stokes eqn. (x - direction)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Select $L \equiv$ **Characteristic length** $V \equiv$ **Charcteristic velocity**

$\Rightarrow L/V \equiv$ Charcteristic time; $\rho V^2 =$ Characteristic pressure

Define $x_0 = x/L$，$y_0 = y/L$，$z_0 = z/L$

$u_0 = u/V$，$v_0 = v/V$，$w_0 = w/V$

$t_0 = t/(L/V)$，$p_0 = p/(\rho V^2)$

⇨ $x = x_0 L$，$y = y_0 L$，$z = z_0 L$

$u = u_0 V$，$v = v_0 V$，$w = w_0 V$

$t = t_0 (L/V)$，$p = p_0 (\rho V^2)$

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代回 N-S eqn.

$$\left(\frac{V^2}{L}\right)\left(\frac{\partial u_0}{\partial t_0}+u_0\frac{\partial u_0}{\partial x_0}+v_0\frac{\partial u_0}{\partial y_0}+w_0\frac{\partial u_0}{\partial z_0}\right)$$

$$=-g_x-\left(\frac{V^2}{L}\right)\frac{\partial p_0}{\partial x_0}+\left(\frac{\nu V}{L^2}\right)\left(\frac{\partial^2 u_0}{\partial x_0^2}+\frac{\partial^2 u_0}{\partial y_0^2}+\frac{\partial^2 u_0}{\partial z_0^2}\right)$$

$$\Rightarrow \boxed{\frac{\partial u_0}{\partial t_0}+u_0\frac{\partial u_0}{\partial x_0}+v_0\frac{\partial u_0}{\partial y_0}+w_0\frac{\partial u_0}{\partial z_0}}$$

$$=-\left(\frac{g_x L}{V^2}\right)-\frac{\partial p_0}{\partial x_0}+\left(\frac{\nu}{VL}\right)\left(\frac{\partial^2 u_0}{\partial x_0^2}+\frac{\partial^2 u_0}{\partial y_0^2}+\frac{\partial^2 u_0}{\partial z_0^2}\right)$$

$$=\boxed{-\left(\frac{1}{Fr^2}\right)-\frac{\partial p_0}{\partial x_0}+\left(\frac{1}{Re}\right)\left(\frac{\partial^2 u_0}{\partial x_0^2}+\frac{\partial^2 u_0}{\partial y_0^2}+\frac{\partial^2 u_0}{\partial z_0^2}\right)}$$

- **Given Fr and Re ⇨ Solution is valid for all scales (prototype and model).**
- **Solution is irrelevant to unit system (solution can be converted back).**