Fourier Analysis MA3019 Midterm Exam 1

National Central University, 2016

Problem 1. Use the Fourier series to show that $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a $2L$-periodic function, and $\widehat{f}_k = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{k\pi x}{L}}\,dx$ be the associated Fourier coefficients. Show that

$$\frac{1}{2L}\int_{-L}^{L} \big|f(x)\big|^2\,dx = \sum_{k=-\infty}^{\infty} \big|\widehat{f}_k\big|^2 .$$

Problem 3. Let $f \in \mathscr{C}(\mathbb{T})$ and $\{\widehat{f}_k\}_{k=-\infty}^{\infty}$ be the Fourier coefficients. Show that if $\sum\limits_{k=-\infty}^{\infty} \big|\widehat{f}_k\big| < \infty$, then $s_n(f,\cdot) \to f$ uniformly on $\mathbb{T}$, where $s_n(f,x) = \sum\limits_{k=-n}^{n} \widehat{f}_k e^{ikx}$.

Problem 4. This problem contributes to another proof of showing that the n-th partial sum of the Fourier series representation $s_n(f,\cdot)$ converges uniformly to f on $\mathbb{T}$ if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for $\frac{1}{2} < \alpha \leqslant 1$. Complete the following.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic such that f is bounded, Riemann integrable over $[-\pi,\pi]$. Show that
$$\widehat{f}_k = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\big(x + \frac{\pi}{k}\big) e^{-ikx}\,dx$$
and hence
$$\widehat{f}_k = \frac{1}{4\pi}\int_{-\pi}^{\pi} \big[f(x) - f\big(x + \frac{\pi}{k}\big)\big] e^{-ikx}\,dx .$$
Therefore, if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies $|\widehat{f}_k| \leqslant \frac{\pi^\alpha \|f\|_{\mathscr{C}^{0,\alpha}(\mathbb{T})}}{2k^\alpha}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic such that f is bounded, Riemann integrable over $[-\pi,\pi]$. Show that
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \big|f(x+h) - f(x-h)\big|^2\,dx = \sum_{k=-\infty}^{\infty} 4\sin^2(kh)|\widehat{f}_k|^2 .$$
Therefore, if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, the Fourier coefficients $\widehat{f}_k$ satisfies
$$\sum_{k=-\infty}^{\infty} \sin^2(kh)|\widehat{f}_k|^2 \leqslant \|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})} 2^{2(\alpha-1)} |h|^{2\alpha} \tag{0.1}$$

3. Let $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$, and $p \in \mathbb{N}$. Show that
$$\sum_{2^{p-1} \leqslant |k| < 2^p} |\widehat{f}_k|^2 \leqslant \frac{\|f\|^2_{\mathscr{C}^{0,\alpha}(\mathbb{T})}\pi^{2\alpha}}{2^{2\alpha p+1}} .$$
Hint: Let $h = \frac{\pi}{2^{p+1}}$ in (0.1).

4. Show that if $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$ for some $\dfrac{1}{2} < \alpha \leqslant 1$, then $\sum\limits_{k=-\infty}^{\infty} |\widehat{f}_k| < \infty$; thus Problem 3 implies that $s_n(f,\cdot) \to f$ uniformly on $\mathbb{T}$.

Problem 5. Let f be a 2π-periodic Lipchitz function. Show that for $n \geqslant 2$,

$$\|f - F_{n+1} \star f\|_{L^\infty(\mathbb{T})} \leqslant \frac{1 + 2\log n}{2n} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} \tag{0.2}$$

and

$$\big\|f - s_n(f,\cdot)\big\|_{L^\infty(\mathbb{T})} \leqslant \frac{2\pi(1+\log n)^2}{n} \|f\|_{\mathscr{C}^{0,1}(\mathbb{T})} \,. \tag{0.3}$$

Hint: For (0.2), apply the estimate

$$F_n(x) \leqslant \min\Big\{\frac{n+1}{2\pi}, \frac{\pi}{2(n+1)x^2}\Big\}$$

in the following inequality:

$$\big|f(x) - F_{n+1} \star f(x)\big| \leqslant \Big[\int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\Big] \big|f(x+y) - f(x)\big| F_{n+1}(y)\, dy$$

with $\delta = \dfrac{\pi}{n+1}$. For (0.3), use (2.8) in the lecture note and note that

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_{L^\infty(\mathbb{T})} \leqslant \|f - F_n \star f\|_{L^\infty(\mathbb{T})} \,.$$

Problem 6. In this problem, we are concerned with the following

Theorem 0.1 (Bernstein)**.** *Suppose that f is a 2π-periodic function such that for some constant C and $\alpha \in (0,1)$,*

$$\inf_{p \in \mathscr{P}_n(\mathbb{T})} \|f - p\|_{L^\infty(\mathbb{T})} \leqslant C n^{-\alpha}$$

for all $n \in \mathbb{N}$. Then $f \in \mathscr{C}^{0,\alpha}(\mathbb{T})$.

Complete the following to prove the theorem.

1. Suppose that there is $p \in \mathscr{P}_n(\mathbb{T})$ such that

   $$\|p'\|_{L^\infty(\mathbb{T})} > n\,, \quad \|p\|_{L^\infty(\mathbb{T})} < 1\,, \quad \text{and} \quad p'(0) = \|p'\|_{L^\infty(\mathbb{T})}.$$

   Choose $\gamma \in \big[-\dfrac{\pi}{n}, \dfrac{\pi}{n}\big]$ such that $\sin(n\gamma) = -p(0)$ and $\cos(n\gamma) > 0$, and define $\alpha_k = \gamma + \dfrac{\pi}{n}\big(k + \dfrac{1}{2}\big)$ for $-n \leqslant k \leqslant n$. Show that the function $r(x) = \sin n(x-\gamma) - p(x)$ has at least one zeros in each interval (α_k, α_{k+1}).

2. Let $s \in \mathbb{Z}$ be such that $0 \in (\alpha_s, \alpha_{s+1})$. Show that r has at least 3 distinct zeros in (α_s, α_{s+1}) by noting that $r'(0) < 0$ and $r(0) = 0$.

3. Combining 1 and 2, show that

   $$\|p'\|_{L^\infty(\mathbb{T})} \leqslant n\|p\|_{L^\infty(\mathbb{T})} \qquad \forall\, p \in \mathscr{P}_n(\mathbb{T}). \tag{0.4}$$

4. Choose $p_n \in \mathscr{P}_n(\mathbb{T})$ such that $\|f - p_n\| \leqslant 2Cn^{-\alpha}$ for $n \in \mathbb{N}$. Define $q_0 = p_1$, and $q_n = p_{2^n} - p_{2^{n-1}}$ for $n \in \mathbb{N}$. Show that $\sum_{n=0}^{\infty} q_n = f$ and the convergence is uniform.

5. Show that $\|q_n\|_{L^\infty(\mathbb{T})} \leqslant 6C2^{-n\alpha}$. As a consequence, show that

$$\big|q_n(x) - q_n(y)\big| \leqslant 6Cn2^{n(1-\alpha)}|x-y| \quad \text{and} \quad \big|q_n(x) - q_n(y)\big| \leqslant 12C2^{-n\alpha}.$$

6. For any $x, y \in \mathbb{T}$ with $|x - y| \leqslant 1$, choose $m \in \mathbb{N}$ such that $2^{-m} \leqslant |x-y| \leqslant 2^{1-m}$. Then use the inequality

$$\big|f(x) - f(y)\big| \leqslant \sum_{n=0}^{m-1} \big|q_n(x) - q_n(y)\big| + \sum_{n=m}^{\infty} \big|q_n(x) - q_n(y)\big|$$

to show that $\big|f(x) - f(y)\big| \leqslant B|x-y|^\alpha$ for some constant $B > 0$.