

Extra Exercise Problem Sets 1

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Problem 1. Determine whether the sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

$$(1) a_n = \frac{\ln n}{\ln(2n)} \quad (2) a_n = \frac{(-1)^{n+1}n}{n + \sqrt{n}} \quad (3) a_n = n \sin \frac{1}{n} \quad (4) a_n = n - \sqrt{n+1}\sqrt{n+3}$$

$$(5) a_n = \sqrt[n]{n^2 + n} \quad (6) a_n = (3^n + 5^n)^{\frac{1}{n}} \quad (7) a_n = \frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$$

$$(8) a_n = \sqrt{n} \ln \left(1 + \frac{1}{n}\right) \quad (9) a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} \quad (10) a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (n+1)!}.$$

Problem 2. Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent. If it is convergent, find its sum.

$$(1) a_n = \frac{1}{1 + (\frac{2}{3})^n} \quad (2) a_n = \ln \left(\frac{n^2 + 1}{2n^2 + 1}\right) \quad (3) a_n = e^{-n} + \frac{1}{n(n+1)}$$

$$(4) a_n = \frac{1}{n^3 - n} \quad (5) a_n = \frac{40n}{(2n-1)^2(2n+1)^2}$$

Problem 3. Find values of x for which the following series converges.

$$(1) \sum_{n=1}^{\infty} (-4)^n (x-5)^n \quad (2) \sum_{n=1}^{\infty} \frac{2^n}{x^n} \quad (3) \sum_{n=1}^{\infty} \frac{\sin^n x}{3^n} \quad (4) \sum_{n=1}^{\infty} e^{nx}.$$

Problem 4. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers.

(1) Show that if $\lim_{n \rightarrow \infty} (a_n + b_n)$ D.N.E. and $\lim_{n \rightarrow \infty} b_n$ converges, then $\lim_{n \rightarrow \infty} a_n$ D.N.E.

(2) Show that if $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Problem 5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of real numbers defined by

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n} \sum_{k=1}^n a_k.$$

(1) Show that if $\lim_{n \rightarrow \infty} a_n = a$ exists, then $\lim_{n \rightarrow \infty} \sigma_n = a$.

(2) Suppose that $\lim_{n \rightarrow \infty} \sigma_n = a$ exists, is it necessary that $\lim_{n \rightarrow \infty} a_n = a$?

Problem 6. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers defined recursively by

$$a_{n+1} = \sqrt{1 + a_n} \quad \forall n \in \mathbb{N} \cup \{0\}, a_0 = 0.$$

Show that $\{a_n\}_{n=1}^{\infty}$ converges and find the limit.

Problem 7. Let $a_n = \left(1 + \frac{1}{n}\right)^n$.

(1) Show that if $0 \leq a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n.$$

(2) Deduce that $b^n[(n + 1)a - nb] < a^{n+1}$.

(3) Use $a = 1 + \frac{1}{n+1}$ and $b = 1 + \frac{1}{n}$ in (2) to show that $\{a_n\}_{n=1}^{\infty}$ is (strictly) increasing.

(4) Use $a = 1$ and $b = 1 + \frac{1}{2n}$ in (2) to show that $a_{2n} < 4$.

(5) Use (3) and (4) to show that $a_n < 4$.

(6) Deduce that $\{a_n\}_{n=1}^{\infty}$ converges.

Problem 8. Let a, b be positive real numbers, $a > b$. Let two sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be given by the recursive relation

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n} \quad \forall n \in \mathbb{N}, \quad a_1 = \frac{a + b}{2}, \quad b_1 = \sqrt{ab}.$$

Complete the following.

(1) Show (by induction) that $a_n > a_{n+1} > b_{n+1} > b_n$ for all $n \in \mathbb{N}$.

(2) Deduce that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converges.

(3) Show that $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ both exist and are identical.

Problem 9. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real number defined by the recursive relation

$$a_{n+1} = \frac{1}{2 + a_n} \quad \forall n \geq 0, \quad a_0 = \frac{1}{2}.$$

Complete the following.

(1) Show that the sequence $\{a_{2n}\}_{n=0}^{\infty}$ is a decreasing sequence; that is, $a_{2n+2} \leq a_{2n}$ for all $n \in \mathbb{N} \cup \{0\}$.

(2) Show that the sequence $\{a_{2n+1}\}_{n=0}^{\infty}$ is an increasing sequence; that is, $a_{2n+3} \geq a_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

(3) Show that $a_{2k+1} \leq a_{2\ell}$ for all $k, \ell \in \mathbb{N} \cup \{0\}$.

(4) Show that the two sequences $\{a_{2n}\}_{n=0}^{\infty}$ and $\{a_{2n+1}\}_{n=0}^{\infty}$ converges to the same limit.

(5) Show that $\{a_n\}_{n=0}^{\infty}$ converges.

Problem 10. The Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$ is a sequence defined recursively by

$$f_1 = 1, \quad f_2 = 1 \quad \text{and} \quad f_{n+2} = f_{n+1} + f_n \quad \forall n \in \mathbb{N}.$$

Show the following.

$$(1) \frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} \text{ for all } n \geq 2.$$

$$(2) \sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 1.$$

$$(3) \sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = 2.$$

Problem 11. Consider the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

(1) Find the partial sum S_1, S_2, S_3 and S_4 . Do you recognize the denominators? Use the pattern to guess a formula for S_n .

(2) Prove your guess by induction.

(3) Show that the given series is convergent, and find the sum.

Problem 12. Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

$$(1) a_n = \frac{1}{n^{1+\frac{1}{n}}} \quad (2) a_n = \ln\left(1 + \frac{1}{n^2}\right) \quad (3) a_n = \frac{2^n + 3^n}{3^n + 4^n} \quad (4) a_n = \tan \frac{1}{n}$$

$$(5) a_n = \sin^n \frac{1}{\sqrt{n}} \quad (6) a_n = \frac{\arctan n}{n^{1.1}} \quad (7) a_n = \left[-\ln\left(e^2 + \frac{1}{n^2}\right)\right]^n$$

$$(8) a_n = \left(1 - \frac{1}{n}\right)^{n^2} \quad (9) a_n = \left(1 + \frac{1}{n}\right)^{-n^2} \quad (10) a_n = \frac{(n!)^2}{(2n)!} \quad (11) a_n = \frac{n! \ln n}{n(n+2)!}$$

$$(12) a_n = \frac{n!}{n^n} \quad (13) a_n = \frac{(-1)^n (3n)!}{n!(n+1)!(n+2)!} \quad (14) a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}$$

$$(15) a_n = (-1)^n (\sqrt{n+\sqrt{n}} - \sqrt{n}) \quad (16) a_n = (-1)^n \frac{(n!)^2 3^n}{(2n+1)!} \quad (17) a_n = \frac{(-1)^n (n!)^n}{n^{n^2}}$$

Problem 13. Find all p and q such that $\sum_{k=2}^{\infty} \frac{(\ln k)^q}{k^p}$ converges.

Problem 14. Find all $p \in \mathbb{R}$ such that the series $\sum_{k=3}^{\infty} \frac{\sin(1/k)}{(\ln k)^p \ln(\ln k)}$ converges. Note that you need to provide the reason for the convergence or divergence of the series for each p .

Problem 15. Show that if $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive terms, then $\sum_{k=1}^{\infty} \sin a_k$ converges.

Problem 16. Let $S = \sum_{k=1}^{\infty} \frac{1}{k^2}$. Euler found that $S = \frac{\pi^2}{6}$ in 1735 AD.

$$(1) \text{ Show that } S = 1 + \sum_{k=1}^{\infty} \frac{1}{n^2(n+1)}.$$

(2) Which of the sums $\sum_{k=1}^{1000000} \frac{1}{k^2}$ or $1 + \sum_{k=1}^{1000} \frac{1}{k^2(k+1)}$ should give a better approximation of S ? Explain your answer.

Hint: (1) $\frac{1}{n^2(n+1)} = \frac{1}{n^2} - \frac{1}{n(n+1)}$.

Problem 17. Find all real numbers x such that $\sum_{k=1}^{\infty} \frac{\cos(kx)}{\ln k}$ converges.

Problem 18. Show by example that $\sum_{k=1}^{\infty} a_k b_k$ may diverge even if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge.

Problem 19. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n, b_n > 0$ for all $n \geq N$. Define

$$c_n = b_n - b_{n+1} \frac{a_{n+1}}{a_n} \quad \forall n \in \mathbb{N}. \quad (0.1)$$

1. Show that if there exists a constant $r > 0$ such that $r < c_n$ for all $n \geq N$, then $\sum_{k=1}^{\infty} a_k$ converges.

Hint: Rewrite (0.1) as $b_n = c_n + \frac{a_{n+1}}{a_n} b_{n+1}$ and then obtain

$$\begin{aligned} b_N &= c_N + \frac{a_{N+1}}{a_N} b_{N+1} = c_N + \frac{a_{N+1}}{a_N} \left(c_{N+1} + \frac{a_{N+2}}{a_{N+1}} b_{N+2} \right) = c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} b_{N+2} \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} \left(c_{N+2} + \frac{a_{N+3}}{a_{N+2}} b_{N+3} \right) = \dots \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} c_{N+2} + \dots + \frac{a_{N+n}}{a_N} c_{N+n} + \frac{a_{N+n+1}}{a_N} b_{N+n+1}. \end{aligned}$$

Use the fact that $0 < r < c_n$ for all $n \geq N$ to conclude that

$$\sum_{k=N}^{N+n} a_k \leq \frac{a_N b_N}{r} \quad \forall n \in \mathbb{N}.$$

Note that then the sequence of partial sum of $\sum_{k=1}^{\infty} a_k$ then is bounded from above (by $\sum_{k=1}^{N-1} a_k + \frac{a_N b_N}{r}$).

2. Show that if $\sum_{k=1}^{\infty} \frac{1}{b_k}$ diverges and $c_n \leq 0$ for all $n \geq N$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Hint: The fact that $c_n \leq 0$ for all $n \geq N$ implies that $b_n a_n \leq b_{n+1} a_{n+1}$ for all $n \geq N$. Use this fact to conclude that

$$\frac{a_N b_N}{b_n} \leq a_n \quad \forall n \geq N$$

and then apply the direct comparison test to conclude that $\sum_{k=1}^{\infty} a_k$ diverges.

Problem 20. Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms, and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$. We know from class that the ratio test fails when this happens, but there are some refined results concerning this particular case.

1. **(Raabe's test):**

(a) If there exists a constant $\mu > 1$ such that $\frac{a_{n+1}}{a_n} < 1 - \frac{\mu}{n}$ for all $n \geq N$, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) If there exists a constant $0 < \mu < 1$ such that $\frac{a_{n+1}}{a_n} > 1 - \frac{\mu}{n}$ for all $n \geq N$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Hint: Consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_n = (n-1)a_n - na_{n+1}$. Then $\sum_{k=1}^{\infty} b_k$ is a telescoping series. For case (a), show that $\{na_{n+1}\}_{n=N}^{\infty}$ is a positive decreasing sequence and then conclude that $\sum_{k=1}^{\infty} b_k$ converges. Note that $b_n \geq (\mu-1)a_n$ for all $n \geq N$. For case (b), show that $\{na_{n+1}\}_{n=N}^{\infty}$ is a positive increasing sequence; thus $a_n \geq \frac{Na_{N+1}}{n-1}$ for all $n \geq N+1$ which implies that $\sum_{k=1}^{\infty} a_k$ diverges.

Remark: 注意到 (a) 說的是如果 $\{a_n\}_{n=1}^{\infty}$ 在某項之後「遞減得夠快」，那麼 $\sum_{k=1}^{\infty} a_k$ 收斂。反之，如果 $\{a_n\}_{n=1}^{\infty}$ 「並非遞減得那麼快」，那麼 $\sum_{k=1}^{\infty} a_k$ 發散。

2. **(Gauss's test):** Suppose that there exist a positive constant $\epsilon > 0$, a constant μ , and a bounded sequence $\{R_n\}_{n=1}^{\infty}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} \quad \text{for all } n \geq N.$$

(a) If $\mu > 1$, then $\sum_{k=1}^{\infty} a_k$ converges. (b) If $\mu \leq 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Hint: Show that if $\mu > 1$ or $\mu < 1$, one can apply Raabe's test to conclude Gauss's test. For the case $\mu = 1$, let $b_n = (n-1) \ln(n-1)$ for $n \geq 2$. Using the second result of Problem 19 to show the divergence of $\sum_{k=1}^{\infty} a_k$ (by showing that c_n defined by (0.1) is non-positive for all large enough n).

Problem 21. Complete the following.

1. Show that $\sum_{k=1}^{\infty} \left(1 - \frac{1}{\sqrt{k}}\right)^k$ converges.

2. Show that $\sum_{k=2}^{\infty} \frac{\log(k+1) - \log k}{(\log k)^2}$ converges.

3. Use Gauss's test to show that both the general harmonic series $\sum_{k=1}^{\infty} \frac{1}{ak+b}$, where $a \neq 0$, and the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverge.

4. Show that $\sum_{k=1}^{\infty} \frac{k!}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$.

5. Test the following "hypergeometric" series for convergence or divergence:

(a)
$$\sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+k-1)} = \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \cdots$$

(b)
$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma \cdot (\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} + \cdots$$

Problem 22. Let $\sum_{k=1}^{\infty} a_k$ be a conditionally convergent series. Show that $\sum_{k=1}^{\infty} [1 + \operatorname{sgn}(a_k)]a_k$ and $\sum_{k=1}^{\infty} [1 - \operatorname{sgn}(a_k)]a_k$ both diverge. Here the sign function sgn is defined by

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

Problem 23. A permutation of a non-empty set A is a one-to-one function from A onto A . Let $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be a permutation of \mathbb{N} .

1. Suppose that $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers. Show that $\{a_{\pi(n)}\}_{n=1}^{\infty}$ is also convergent; that is, show that if $\{b_n\}_{n=1}^{\infty}$ is a sequence defined by $b_n = a_{\pi(n)}$, then $\{b_n\}_{n=1}^{\infty}$ also converges.

2. Suppose that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum_{k=1}^{\infty} a_{\pi(k)}$ is also absolutely convergent, and

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\pi(k)}.$$

3. Suppose that $\sum_{k=1}^{\infty} a_k$ is conditionally convergent. Show that for each $r \in \mathbb{R}$, there exists a permutation $\pi : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} a_{\pi(k)} = r.$$