三月十二日課程補充參考資料

以下以兩個與課堂（影片）上不同的方式證明 Taylor 定理。我們首先將 Taylor 定理以如下方式重述：

> 若 $f:(a,b)\to\mathbb{R}$ 為 $(n+1)$ 次可微，則對所有 (a,b) 中相異兩點 c 和 x，存在一個界於 c 和 x 間的點 ξ 使得
> $$\frac{f(x)-\sum\limits_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k}{(x-c)^{n+1}}=\frac{f^{(n+1)}(\xi)}{(n+1)!}\,. \tag{$\star$}$$

與之前相同，我們定義餘項 $R_n(x)=f(x)-\sum\limits_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k$.

Proof 1: In this proof we prove ($\star$) by induction.

(i) **Case** $n=0$: In this case $f:(a,b)$ is differentiable. Therefore, the mean value theorem shows that there exists ξ between c and x such that

$$\frac{f(x)-f(c)}{x-c}=f'(\xi)$$

which shows that ($\star$) holds for $n=0$.

(ii) **Case** $n=m-1$: Assume that ($\star$) holds for the case $n=m-1$ for some $m\in\mathbb{N}$.

(iii) **Case** $n=m$: Suppose that $f:(a,b)\to\mathbb{R}$ is $(m+1)$-times differentiable. Then f' is m-times differentiable on (a,b); thus (ii) implies that

> for all $c,x\in(a,b)$ satisfying $c\neq x$ there exists ξ such that
> $$\frac{f'(x)-\sum\limits_{k=0}^{m-1}\frac{f^{(k+1)}(c)}{k!}(x-c)^k}{(x-c)^{n+1}}=\frac{f^{(m+1)}(\xi)}{m!}\,. \tag{$\star\star$}$$

Let $F(x)=f(x)-\sum\limits_{k=0}^{m}\frac{f^{(k)}(c)}{k!}(x-c)^k$ and $G(x)=(x-c)^{m+1}$. By Cauchy MVT, there exists x_1 between c and x such that

$$\frac{F(x)-F(c)}{G(x)-G(c)}=\frac{F'(x_1)}{G'(x_1)}\,. \tag{0.1}$$

Since

$$\begin{aligned}F'(x)&=f'(x)-\sum_{k=0}^{m}\frac{f^{(k)}(c)}{k!}k(x-c)^{k-1}=f'(x)-\sum_{k=1}^{m}\frac{f^{(k)}(c)}{(k-1)!}(x-c)^{k-1}\\&=f'(x)-\sum_{k=0}^{m-1}\frac{f^{(k+1)}(c)}{k!}(x-c)^k\,,\end{aligned}$$

by ($\star\star$) there exists ξ between c and x_1 such that

$$\frac{F'(x_1)}{G'(x_1)}=\frac{f'(x_1)-\sum\limits_{k=0}^{m-1}\frac{f^{(k+1)}(c)}{k!}(x_1-c)^k}{(m+1)(x_1-c)^m}=\frac{f^{(m+1)}(\xi)}{(m+1)!}\,. \tag{0.2}$$

Combining (0.1) and (0.2), we conclude that ($\star$) holds for $n=m$.

By induction, $(\star)$ holds for all $n \in \mathbb{N} \cup \{0\}$.

Proof 2: In this proof we establish $(\star)$ by treating c as a variable but viewing x is a fixed number. Define

$$F(z) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x-z)^k \quad \text{and} \quad G(z) = (x-z)^{n+1}.$$

Then

$$\begin{aligned}
F'(z) &= -\sum_{k=0}^{n} \frac{d}{dz}\Big[\frac{f^{(k)}(z)}{k!}(x-z)^k\Big] = -\sum_{k=0}^{n}\Big[\frac{f^{(k+1)}(z)}{k!}(x-z)^k + \frac{f^{(k)}(z)}{k!}k(x-z)^{k-1}(-1)\Big] \\
&= -\sum_{k=0}^{n} \frac{f^{(k+1)}(z)}{k!}(x-z)^k + \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}k(x-z)^{k-1} \\
&= -\sum_{k=0}^{n} \frac{f^{(k+1)}(z)}{k!}(x-z)^k + \sum_{k=1}^{n} \frac{f^{(k)}(z)}{(k-1)!}(x-z)^{k-1} \\
&= -\sum_{k=0}^{n} \frac{f^{(k+1)}(z)}{k!}(x-z)^k + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(z)}{k!}(x-z)^{k} = -\frac{f^{(n+1)}(z)}{n!}(x-z)^n
\end{aligned}$$

and

$$G'(z) = -(n+1)(x-z)^n.$$

Let $I = (\min\{c,x\}, \max\{c,x\})$ and $\bar{I} = [\min\{c,x\}, \max\{c,x\}]$. Then $F, G : \bar{I} \to \mathbb{R}$ are continuous and $F, G : I \to \mathbb{R}$ are differentiable. Moreover, $G'(z) \neq 0$ for all $z \in I$. Therefore, the Cauchy MVT implies that there exists ξ between c and x such that

$$\frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F'(\xi)}{G'(\xi)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}.$$

We then conclude $(\star)$ from the fact that $F(x) = G(x) = 0$ and $F(c) = R_n(x)$. □

以下為我們在課堂上展示的 matalb code。

```
≫ dx = 2*pi/100000;
≫ x = dx:dx:2*pi-dx;
≫ y = zeros(1,length(x));
≫ N = 100;
≫ for k=1:N
≫     y = y + sin(k*x)/k;
≫ end
≫ plot(x,y,'b');
```

大家可以改 N 去看不同的 partial sum 會是怎樣的結果。