Exercise Problem Sets 9

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Problem 1. Show that the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1\,.$$

Problem 2. Show that the equation of the tangent plane to the elliptic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}\,.$$

Problem 3. Let f be a differentiable function and consider the surface $z = xf\big(\frac{y}{x}\big)$. Show that the tangent plane at any point (x_0, y_0, z_0) on the surface passes through the origin.

Problem 4. Prove that the angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) satisfies

$$\cos\theta = \frac{1}{\sqrt{f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2 + 1}}\,.$$

Problem 5. In the following problems, find all relative extrema and saddle points of the function. Use the Second Partials Test when applicable.

(1) $f(x, y) = x^2 - xy - y^2 - 3x - y$ (2) $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1$

(3) $f(x, y) = xy - 2x - 2y - x^2 - y^2$ (4) $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

(5) $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$ (6) $f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$

(7) $f(x, y) = \ln(x + y) + x^2 - y$ (8) $f(x, y) = 2\ln x + \ln y - 4x - y$

(9) $f(x, y) = xy\exp\big(-\frac{x^2 + y^2}{2}\big)$ (10) $f(x, y) = xy + e^{-xy}$

(11) $f(x, y) = (x^2 + y^2)e^{-x}$ (12) $f(x, y) = \Big(\frac{1}{2} - x^2 + y^2\Big)\exp(1 - x^2 - y^2)$

Problem 6. In the following problems, find the absolute extrema of the function over the region R (which contains boundaries).

(1) $f(x, y) = x^2 + xy$, and $R = \big\{(x, y)\,\big|\,|x| \leqslant 2, |y| \leqslant 1\big\}$

(2) $f(x, y) = 2x - 2xy + y^2$, and R is the region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 1$.

(3) $f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}$, and $R = \big\{(x, y)\,\big|\,0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\big\}$.

(4) $f(x,y) = xy^2$, and $R = \{(x,y) \,|\, x \geqslant 0, y \geqslant 0, x^2 + y^2 \leqslant 3\}$.

(5) $f(x,y) = 2x^3 + y^4$, and $R = \{(x,y) \,|\, x^2 + y^2 \leqslant 1\}$.

Problem 7. Show that $f(x,y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that the discriminant $f_{xx}f_{yy} - f_{xy}^2 = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point

Problem 8. Show that $f(x,y) = x^2 ye^{-x^2-y^2}$ has maximum values at $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and minimum values at $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$. Show also that f has infinitely many other critical points and the discriminant $f_{xx}f_{yy} - f_{xy}^2 = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

Problem 9. Find two numbers a and b with $a \leqslant b$ such that

$$\int_a^b \sqrt[3]{24 - 2x - x^2}\, dx$$

has its largest value.