Exercise Problem Sets 7

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Problem 1. Let $f, g : (a, b) \to \mathbb{R}$ be real-valued function, $h(x, y) = f(x)g(y)$, and $c, d \in (a, b)$. Show that if f is differentiable at c and g is differentiable at d, then h is differentiable at (c, d).

Problem 2. In the following, show that both $f_x(0, 0)$ and $f_y(0, 0)$ both exist but that f is not differentiable at $(0, 0)$.

(1) $f(x,y) = \begin{cases} \dfrac{5x^2y}{x^3+y^3} & \text{if } x^3+y^3 \neq 0\,, \\ 0 & \text{if } x^3+y^3 = 0\,. \end{cases}$

(2) $f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

(3) $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

(4) $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

Problem 3. Show that the function $f(x, y) = \sqrt{x^2 + y^2} \sin \sqrt{x^2 + y^2}$ is differentiable at $(0, 0)$.

Problem 4. Investigate the differentiability of the following functions at the point $(0, 0)$.

(1) $f(x, y) = \sqrt[3]{x} \cos y$.

(2) $f(x, y) = \sqrt{|xy|}$.

(3) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(4) $f(x,y) = \begin{cases} \dfrac{xy}{x+y^2} & \text{if } x+y^2 \neq 0\,, \\ 0 & \text{if } x+y^2 = 0 \end{cases}$

(5) $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

Problem 5. Let $R \subseteq \mathbb{R}^2$ be an open region, and $f : R \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are bounded on R; that is, there exists a real number $M > 0$ such that

$$\left|\frac{\partial f}{\partial x}(x,y)\right|, \left|\frac{\partial f}{\partial y}(x,y)\right| \leqslant M \qquad \forall\, (x,y) \in R\,.$$

Show that f is continuous on U.

Hint: Make use of the mean value theorem.

Proof. Let $(a,b) \in R$ be given. Since R is open, there exists $\delta > 0$ such that the open disk $D((a,b),\delta) \subseteq R$.

For $(x,y) \in D((a,b),\delta)$, we have

$$\big|f(x,y) - f(a,b)\big| = \big|f(x,y) - f(a,y) + f(a,y) - f(a,b)\big| \leqslant \big|f(x,y) - f(a,y)\big| + \big|f(a,y) - f(a,b)\big| .$$

so the mean value theorem shows that

1. there exists ξ between x and a such that

$$\big|f(x,y) - f(a,y)\big| = \big|f_x(\xi,y)(x-a)\big| \leqslant M|x-a| ;$$

2. there exists η between y and b such that

$$\big|f(a,y) - f(a,b)\big| = \big|f_y(a,\eta)(y-b)\big| \leqslant M|y-b| .$$

Therefore,

$$\big|f(x,y) - f(a,b)\big| \leqslant M\big[|x-a| + |y-b|\big] \qquad \forall\,(x,y) \in D((a,b),\delta) .$$

By the Squeeze Theorem,

$$\lim_{(x,y)\to(a,b)} \big|f(x,y) - f(a,b)\big| = 0 ;$$

thus $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ which shows that f is continuous at (a,b). □

Problem 6. Let $R \subseteq \mathbb{R}^n$ be an open disk, and $f : R \to \mathbb{R}$ be a function such that $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial y}(x,y) = 0$ for all $(x,y) \in R$. Show that f is constant in R.

Proof. From Problem 5, we find that

$$\big|f(x,y) - f(0,0)\big| \leqslant 0\big[|x-0| + |y-0|\big] = 0 \qquad \forall\,(x,y) \in R ;$$

thus $f(x,y) = f(0,0)$ for all $(x,y) \in R$. This shows that f is constant. □

Problem 7. Let $(a,b) \subseteq \mathbb{R}$ be an open interval, and for each $1 \leqslant i,j \leqslant n$, $a_{ij} : (a,b) \to \mathbb{R}$ be differentiable functions. Define $A = [a_{ij}]$ and $J = \det(A)$. Show that

$$\frac{\partial J}{\partial x} = \operatorname{tr}\Big(\operatorname{Adj}(A)\frac{\partial A}{\partial x}\Big) ,$$

where for a square matrix $M = [m_{ij}]$, $\operatorname{tr}(M)$ denotes the trace of M, $\operatorname{Adj}(M)$ denotes the adjoint matrix of M, and $\dfrac{\partial M}{\partial x}$ denotes the matrix whose (i,j)-th entry is given by $\dfrac{\partial m_{ij}}{\partial x}$.

Hint: Show that

$$\frac{\partial J}{\partial x} = \begin{vmatrix} \frac{\partial a_{11}}{\partial x} & a_{12} & \cdots & a_{1n} \\ \frac{\partial a_{21}}{\partial x} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ \frac{\partial a_{n1}}{\partial x} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \frac{\partial a_{12}}{\partial x} & a_{13} & \cdots & a_{1n} \\ a_{21} & \frac{\partial a_{22}}{\partial x} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & \frac{\partial a_{n2}}{\partial x} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & \cdots & a_{(n-1)1} & \frac{\partial a_{1n}}{\partial x} \\ a_{21} & \cdots & a_{(n-1)2} & \frac{\partial a_{2n}}{\partial x} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{(n-1)n} & \frac{\partial a_{n1}}{\partial x} \end{vmatrix}$$

and rewrite this identity in the form which is asked to prove. You can also show the differentiation formula by applying the chain rule to the composite function $F \circ g$ of maps $g : U \to \mathbb{R}^{n^2}$ and $F : \mathbb{R}^{n^2} \to \mathbb{R}$ defined by $g(x) = \big(a_{11}(x), a_{12}(x), \cdots, a_{nn}(x)\big)$ and $F(a_{11}, \cdots, a_{nn}) = \det\big([a_{ij}]\big)$. Check first what $\dfrac{\partial F}{\partial a_{ij}}$ is.

Proof. Let $A = [a_{ij}]$, and $C = [c_{ij}]$ be the cofactor matrix of A; that is,

$$c_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}.$$

In other words, the (i,j)-entry of C is $(-1)^{i+j}$ multiplied by the determinant of the $(n-1)\times(n-1)$ matrix obtained by removing the i-th row and j-th column of A. Then the (i,j)-entry of the adjoint matrix of A is c_{ji}; that is,

$$\mathrm{Adj}(A) = C^{\mathrm{T}}.$$

By the property (cofactor expansion) of the determinant,

$$\det(A) = \sum_{k=1}^{n} a_{ik}c_{ik} \qquad \text{for all } 1 \leqslant i \leqslant n.$$

Since the computation of c_{ik} does not involve the knowledge of a_{i1}, a_{i2}, $\cdots$, a_{in}, we find that

$$\frac{\partial c_{ik}}{\partial a_{ij}} = 0 \qquad \text{for all } 1 \leqslant j,k \leqslant n.$$

Therefore, the product rule implies that

$$\frac{\partial \det(A)}{\partial a_{ij}} = \sum_{k=1}^{n} \Big[\frac{\partial a_{ik}}{\partial a_{ij}} c_{ik} + a_{ik}\frac{\partial c_{ik}}{\partial a_{ij}}\Big] = \sum_{k=1}^{n} \delta_{kj}c_{ik}\,,$$

where $\delta_{\cdot\cdot}$ is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Therefore, $\dfrac{\partial \det(A)}{\partial a_{ij}} = c_{ij}$.

Now suppose that each a_{ij} is a differentiable function defined on (a,b), and

$$J(x) = \begin{vmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{vmatrix}$$

Then the chain rule shows that

$$J'(x) = \sum_{i,j=1}^{n} \frac{\partial \det(A)}{\partial a_{ij}} \frac{da_{ij}}{dx}(x) = \sum_{i,j=1}^{n} c_{ij}(x) a'_{ij}(x)\,,$$

where $C(x) = [c_{ij}(x)]$ is the cofactor matrix of $A(x) = [a_{ij}(x)]$. Let $D(x) = [d_{ij}(x)]$ be the adjoint matrix of $A(x)$. Then $d_{ij}(x) = c_{ji}(x)$. Note that for each $1 \leqslant j, k \leqslant n$,

$$\sum_{i=1}^{n} c_{ij}(x) a'_{ik}(x) = \sum_{i=1}^{n} d_{ji}(x) a'_{ik}(x) = \text{ the } (j,k)\text{-entry of } D(x)A'(x).$$

Therefore,

$$J'(x) = \sum_{i,j=1}^{n} c_{ij}(x) a'_{ij}(x) = \sum_{j=1}^{n} \text{ the } (j,j)\text{-entry of } D(x)A'(x)$$

which shows that $J'(x) = \operatorname{tr}(D(x)A(x))$, as desired. □