Exercise Problem Sets 6

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Problem 1. In the following sub-problems, find the limit if it exists or explain why it does not exist.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y}$ (2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$ (3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

(4) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (5) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (6) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

(7) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$ (8) $\displaystyle\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$ (9) $\displaystyle\lim_{(x,y)\to(0,0)} x\cos\frac{1}{y}$

(10) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$ (11) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+zx}{x^2+y^2+z^2}$

(12) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$ (13) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \arctan\frac{1}{x^2+y^2+z^2}$

Problem 2. Discuss the continuity of the functions given below.

1. $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy} & \text{if } xy \neq 0\,, \\ 1 & \text{if } xy = 0\,. \end{cases}$

2. $f(x,y) = \begin{cases} \dfrac{e^{-x^2-y^2}-1}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 1 & \text{if } (x,y) = (0,0)\,. \end{cases}$

3. $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$

Solution. 3. Since $|\sin x| \leqslant |x|$ for all $x \in \mathbb{R}$, we find that if $(x,y) \neq (0,0)$,

$$0 \leqslant \big|f(x,y)\big| \leqslant \left|\frac{x^3+y^4}{x^2+y^2}\right| \leqslant |x| + y^2\,.$$

Since $\displaystyle\lim_{(x,y)\to(0,0)} |x|+y^2 = 0$, the Squeeze Theorem implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

which shows that f is continuous at $(0,0)$. On the other hand, if $(a,b) \neq (0,0)$, then the continuity of the sine function and polynomials shows that

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{\sin(x^3+y^4)}{x^2+y^2} = \frac{\displaystyle\lim_{(x,y)\to(a,b)} \sin(x^3+y^4)}{\displaystyle\lim_{(x,y)\to(a,b)} (x^2+y^2)} = \frac{\sin(a^3+3^4)}{a^2+b^2} = f(a,b)\,;$$

thus f is continuous on $\mathbb{R}^2$. □

Problem 3. Let $f(x,y) = \begin{cases} 0 & \text{if } y \leqslant 0 \text{ or } y \geqslant x^4\,, \\ 1 & \text{if } 0 < y < x^4\,. \end{cases}$

1. Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any path through $(0,0)$ of the form $y = mx^\alpha$ with $0 < \alpha < 4$.

2. Show that f is discontinuous on two entire curves.

Problem 4. Find $\dfrac{\partial}{\partial x}\Big|_{(x,y,z)=(\ln 4,\ln 9,2)} \sum\limits_{n=0}^{\infty} \dfrac{(x+y)^n}{n! z^n}$. Do not write the answer in terms of an infinite sum.

Solution. Note that for each $x, y \in \mathbb{R}$ and $z \neq 0$, we have

$$\sum_{n=0}^{\infty} \frac{(x+y)^n}{n! z^n} = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{x+y}{z}\Big)^n = \exp\Big(\frac{x+y}{z}\Big)\,.$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial x}\Big|_{(x,y,z)=(\ln 4,\ln 9,2)} \sum_{n=0}^{\infty} \frac{(x+y)^n}{n! z^n} &= \frac{\partial}{\partial x}\Big|_{(x,y,z)=(\ln 4,\ln 9,2)} \exp\Big(\frac{x+y}{z}\Big) \\ &= \exp\Big(\frac{x+y}{z}\Big)\frac{1}{z}\Big|_{(x,y,z)=(\ln 4,\ln 9,2)} = \frac{1}{2}\exp\Big(\frac{\ln 4 + \ln 9}{2}\Big) \end{aligned}$$

so the fact that $\dfrac{\ln 4 + \ln 9}{2} = \ln 6$ shows that $\dfrac{\partial}{\partial x}\Big|_{(x,y,z)=(\ln 4,\ln 9,2)} \sum\limits_{n=0}^{\infty} \dfrac{(x+y)^n}{n! z^n} = 3.$ □

Problem 5. Let $f(x,y) = (x^2+y^2)^{\frac{2}{3}}$. Find the partial derivative $\dfrac{\partial f}{\partial x}$.

Problem 6. Let $f(x,y,z) = xy^2z^3 + \arcsin(x\sqrt{z})$. Find f_{xzy} in the region $\{(x,y,z)\,\big|\,|x^2 z| < 1\}$.

Problem 7. Let $\vec{a} = (a_1, a_2, \cdots, a_n)$ be a unit vector, $\vec{x} = (x_1, x_2, \cdots, x_n)$, and $f(x_1, x_2, \cdots, x_n) = \exp(\vec{a}\cdot\vec{x})$. Show that

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = f\,.$$

Problem 8. Let $f(x,y) = x(x^2+y^2)^{-\frac{3}{2}} e^{\sin(x^2 y)}$. Find $f_x(1,0)$.

Problem 9. Let $f(x,y) = \displaystyle\int_1^y \frac{dt}{\sqrt{1-x^3t^3}}$. Show that

$$f_x(x,y) = \int_1^y \Big(\frac{\partial}{\partial x}\frac{1}{\sqrt{1-x^3t^3}}\Big)\,dt$$

in the region $\{(x,y)\,\big|\,x < 1, y > 1 \text{ and } xy < 1\}$.

Solution. By the substitution of variable $s = xt$, we find that

$$f(x,y) = \int_x^{xy} \frac{ds}{x\sqrt{1-s^3}} = \frac{1}{x}\int_x^{xy} \frac{dt}{\sqrt{1-t^3}}\,.$$

Therefore, by the FTOC we obtain that

$$\begin{aligned} f_x(x,y) &= -\frac{1}{x^2}\int_x^{xy}\frac{dt}{\sqrt{1-t^3}} + \frac{y}{\sqrt{1-x^3y^3}} - \frac{1}{\sqrt{1-x^3}} \\ &= -\frac{1}{x}\int_1^{y}\frac{dt}{\sqrt{1-x^3t^3}} + \frac{y}{\sqrt{1-x^3y^3}} - \frac{1}{\sqrt{1-x^3}}\,. \end{aligned}$$

On the other hand, since

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{1-x^3t^3}} = \frac{3x^2t^3}{2(1-x^3t^3)^{3/2}} \quad \text{and} \quad \frac{\partial}{\partial t}\frac{1}{\sqrt{1-x^3t^3}} = \frac{3x^3t^2}{2(1-x^3t^3)^{3/2}}\,,$$

we have

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{1-x^3t^3}} = \frac{t}{x}\frac{\partial}{\partial t}\frac{1}{\sqrt{1-x^3t^3}}$$

so that integrating by parts shows that

$$\begin{aligned} \int_1^y \Big(\frac{\partial}{\partial x}\frac{1}{\sqrt{1-x^3t^3}}\Big)\,dt &= \int_1^y \Big(\frac{t}{x}\frac{\partial}{\partial t}\frac{1}{\sqrt{1-x^3t^3}}\Big)\,dt = \frac{t}{x}\frac{1}{\sqrt{1-x^3t^3}}\Big|_{t=1}^{t=y} - \int_1^y \frac{1}{\sqrt{1-x^3t^3}}\frac{\partial}{\partial t}\frac{t}{x}\,dt \\ &= \frac{y}{x\sqrt{1-x^3y^3}} - \frac{1}{x\sqrt{1-x^3}} - \frac{1}{x}\int_1^y \frac{dt}{\sqrt{1-x^3t^3}} \end{aligned}$$

which is identical to $f_x(x,y)$ computed above. □

Problem 10. The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure P, and volume V is $PV = mRT$, where R is the gas constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1\,.$$

Proof. Since $PV = mRT$,

$$\frac{\partial P}{\partial V} = -\frac{mRT}{V^2}\,, \quad \frac{\partial V}{\partial T} = \frac{mR}{P} \quad \text{and} \quad \frac{\partial T}{\partial P} = \frac{V}{mR}\,.$$

Therefore,

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -\frac{mRT}{V^2}\cdot\frac{mR}{P}\cdot\frac{V}{mR} = -\frac{mRT}{PV} = -1$$

which conclude the proof. □

Problem 11. The total resistance R produced by three conductors with resistances R_1, R_2, R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\,.$$

Find $\dfrac{\partial R}{\partial R_1}$ by directly taking the partial derivative of the equation above.

Solution. Taking the partial derivative of the equation w.r.t. R_1, we obtain that

$$\frac{\partial}{\partial R_1}\frac{1}{R} = \frac{\partial}{\partial R_1}\Big(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\Big) = -\frac{1}{R_1^2}\,;$$

thus the implicit differentiation shows that

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}\,.$$

Therefore, $\dfrac{\partial R}{\partial R_1} = \dfrac{R^2}{R_1^2}$. □

Problem 12. Find the value of $\dfrac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation

$$xy + z^3 x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Problem 13. Find the value of $\dfrac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation

$$xz + y\ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Problem 14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f_x(a,b)$ and $f_y(a,b)$ exists. Suppose that $c = f(a,b)$.

1. Using the geometric meaning of partial derivatives, explain what the vectors $(1,0,f_x(a,b))$ and $(0,1,f_y(a,b))$ mean.
2. Suppose that you know that there is a tangent plane (which we have not talked about, but you can roughly imagine what it is) of the graph of f at (a,b,c). What should the equation of the tangent plane be?

Problem 15. Define

$$f(x,y) = \begin{cases} x^2 \arctan\dfrac{y}{x} - y^2 \arctan\dfrac{x}{y} & \text{if } x, y \neq 0\,, \\ 0 & \text{if } x = 0 \text{ or } y = 0\,. \end{cases}$$

Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.