Exercise Problem Sets 3

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Problem 1. The second Taylor polynomial for a twice-differentiable function f at $x = c$ is called the quadratic approximation of f at $x = c$. Find the quadratic approximate of the following functions at $x = 0$.

(1) $f(x) = \ln \cos x$ (2) $f(x) = e^{\sin x}$ (3) $f(x) = \tan x$ (4) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

(5) $f(x) = e^x \sin^2 x$ (6) $f(x) = e^x \ln(1+x)$ (7) $f(x) = (\arctan x)^2$

Problem 2. Let f have derivatives through order n at $x = c$. Show that the n-th Taylor polynomial for f at c and its first n derivatives have the same values that f and its first n derivatives have at $x = c$.

Problem 3. Suppose that f is differentiable on an interval centered at $x = c$ and that $g(x) = b_0 + b_1(x-c) + \cdots + b_n(x-c)^n$ is a polynomial of degree n with constant coefficients b_0, b_1, $\cdots$, b_n. Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

1. $E(c) = 0$ (which means "the approximation error is zero at $x = c$");
2. $\displaystyle\lim_{x\to c} \frac{E(x)}{(x-c)^n} = 0$ (which means "the error is negligible when compared to $(x-c)^n$),

then g is the n-th Taylor polynomial for f at c. **Thus, the Taylor polynomial P_n is the only polynomial of degree less than or equal to n whose error is both zero at $x = c$ and negligible when compared with $(x-c)^n$.**

Problem 4. Show that if p is an polynomial of degree n, then

$$p(x+1) = \sum_{k=0}^{n} \frac{p^{(k)}(x)}{k!}\,.$$

Problem 5. In Chapter 3 we considered Newton's method for approximating a root/zero r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations x_2, x_3, $\cdots$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \forall\, n \geqslant 1\,.$$

Show that if f'' exists on an interval I containing r, x_n, and x_{n+1}, and $\left|f''(x)\right| \leqslant M$ and $\left|f'(x)\right| \geqslant K$ for all $x \in I$, then

$$|x_{n+1} - r| \leqslant \frac{M}{2K}|x_n - r|^2$$

This means that if x_n is accurate to d decimal places, then x_{n+1} is accurate to about $2d$ decimal places. More precisely, if the error at stage n is at most 10^{-m}, then the error at stage $n+1$ is at most $\dfrac{M}{2K}10^{-2m}$.

Hint: Apply Taylor's Theorem to write $f(r) = P_2(r) + R_2(r)$, where P_2 is the second Taylor polynomial for f at x_n.

Proof. By Taylor's theorem, there exists ξ between r and x_n such that

$$0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{f''(\xi)}{2}(x_n - r)^2 .$$

Therefore,

$$x_n - r - \frac{f'(x_n)}{f(x_n)} = \frac{f''(\xi)}{2}(x_n - r)^2 ;$$

thus by the iterative relation we obtain that

$$|x_{n+1} - r| = \left| x_n - \frac{f'(x_n)}{f(x_n)} - r \right| = \left| \frac{f''(\xi)}{2}(x_n - r)^2 \right| \leqslant \frac{M}{2K}|x_n - r|^2 . \qquad \square$$

Problem 6. Consider a function f with continuous first and second derivatives at $x = c$. Prove that if f has a relative maximum at $x = c$, then the second Taylor polynomial centered at $x = c$ also has a relative maximum at $x = c$.

Problem 7. Let $f : (a, b) \to \mathbb{R}$ be $(n+1)$-times differentiable, and $c \in (a, b)$. In this problem you are ask to derive the remaind associated with the n-th Taylor polynomial for f at c in Schlomilch-Roche form:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!p}(x - c)^p (x - \xi)^{n+1-p} . \qquad (\star)$$

Suppose that $f : (a, b) \to \mathbb{R}$ is $(n + 1)$-times differentiable. For a fixed $x \in (a, b)$, define

$$\varphi(z) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x - z)^k .$$

Note that $\varphi(c) = R_n(x)$. Complete the following.

1. Show that $\varphi'(z) = -\dfrac{f^{(n+1)}(z)}{n!}(x - z)^n$.
2. Apply the Cauchy mean value theorem to the two functions $\varphi(z)$ and $\psi(z) \equiv (x - z)^p$ for some $p \in \{1, 2, \cdots, n\}$; that is,

   $$\frac{\varphi(x) - \varphi(c)}{\psi(x) - \psi(c)} = \frac{\varphi'(\xi)}{\psi'(\xi)} \qquad \text{for some } \xi \text{ between } c \text{ and } x,$$

   to show $(\star)$.
3. Use $(\star)$ to show that

   $$\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \qquad \forall\, x \in (-1, 1] . \qquad (\star\star)$$

Remark: The remainder in Schlomilch-Roche form with $p = 1$ is called Cauchy remainder, and Lagrange remainder is obtained by letting $p = n + 1$ in $(\star)$.

Proof. 1. We compute the derivative of φ as follows:

$$\begin{aligned}\varphi'(z) &= -\sum_{k=0}^{n} \frac{d}{dz}\Big[\frac{f^{(k)}(z)}{k!}(x-z)^k\Big] = -\sum_{k=0}^{n}\Big[\frac{f^{(k+1)}(z)}{k!}(x-z)^k + \frac{f^{(k)}(z)}{k!}k(x-z)^{k-1}(-1)\Big]\\ &= -\sum_{k=0}^{n}\frac{f^{(k+1)}(z)}{k!}(x-z)^k + \sum_{k=0}^{n}\frac{f^{(k)}(z)}{k!}k(x-z)^{k-1}\\ &= -\sum_{k=0}^{n}\frac{f^{(k+1)}(z)}{k!}(x-z)^k + \sum_{k=1}^{n}\frac{f^{(k)}(z)}{(k-1)!}(x-z)^{k-1}\\ &= -\sum_{k=0}^{n}\frac{f^{(k+1)}(z)}{k!}(x-z)^k + \sum_{k=0}^{n-1}\frac{f^{(k+1)}(z)}{k!}(x-z)^k = -\frac{f^{(n+1)}(z)}{n!}(x-z)^n\end{aligned}$$

2. Let $I = (\min\{c,x\}, \max\{c,x\})$ and $\bar{I} = [\min\{c,x\}, \max\{c,x\}]$. Then $\varphi, \psi : \bar{I} \to \mathbb{R}$ are continuous and $\varphi, \psi : I \to \mathbb{R}$ are differentiable. Moreover, $\psi'(z) = -p(x-z)^{p-1}$ so that $\psi'(z) \neq 0$ for all $z \in I$. Therefore, the Cauchy MVT implies that there exists ξ between c and x such that

$$\frac{\varphi(x)-\varphi(c)}{\psi(x)-\psi(c)} = \frac{\varphi'(\xi)}{\psi'(\xi)} = \frac{-\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{-p(x-\xi)^{p-1}} = \frac{f^{(n+1)}(\xi)}{n!p}(x-\xi)^{n+1-p}.$$

Since $\varphi(x) = \psi(x) = 0$, we find that

$$R_n(x) = \varphi(c) = \frac{f^{(n+1)}(\xi)}{n!p}(x-\xi)^{n+1-p}\psi(c) = \frac{f^{(n+1)}(\xi)}{n!p}(x-\xi)^{n+1-p}(x-c)^p.$$

3. Let $f(x) = \ln(1+x)$. Then

$$f^{(n+1)}(x) = (-1)^n n!(1+x)^{-(n+1)}. \qquad (\diamond)$$

(a) **the case** $x \in (0,1]$: using $(\star)$ (with $c=0$ and $p=n+1$) and $(\diamond)$ we find that

$$R_n(x) = \frac{(-1)^n n!(1+\xi)^{-(n+1)}}{n!(n+1)}x^{n+1} = \frac{(-1)^n}{n+1}\Big(\frac{x}{1+\xi}\Big)^n.$$

Since $0 < \xi < x \leqslant 1$, we have $\left|\frac{x}{1+\xi}\right| \leqslant 1$; thus

$$|R_n(x)| \leqslant \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty.$$

Therefore, identity $(\star\star)$ holds for $x \in (0,1]$.

(b) **the case** $x \in (-1,0)$: using $(\star)$ (with $c=0$ and $p=1$) and $(\diamond)$ we find that

$$R_n(x) = \frac{(-1)^n n!(1+\xi)^{-(n+1)}}{n!}x(x-\xi)^n = (-1)^n\frac{x}{1+\xi}\Big(\frac{x-\xi}{1+\xi}\Big)^n.$$

Since $-1 < x < \xi < 0$, we have $\left|\frac{x-\xi}{1+\xi}\right| \leqslant |x|$; thus

$$|R_n(x)| \leqslant \left|\frac{x}{1+\xi}\right||x|^n \to 0 \quad \text{as } n \to \infty.$$

Therefore, identity $(\star\star)$ holds for $x \in (-1,0)$.

(c) Clearly, identity $(\star\star)$ holds for $x = 0$.

Combining the three cases above, we conclude that identity $(\star\star)$ holds for $x \in (-1,1]$. □