Exercise Problem Sets 9

Nov. 17. 2023

Problem 1. Find the following definite integrals (without using any further techniques of integrations).

1. $\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx.$
2. $\int_0^{\frac{\pi}{3}} (\cos x + \sec x)^2 \, dx.$
3. $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x} \, dx.$
4. $\int_0^{\frac{\pi}{6}} (\sec x + \tan x)^2 \, dx.$
5. $\int_0^{\pi} (\cos x + |\cos x|) \, dx.$
6. $\int_0^{\frac{3\pi}{2}} |\sin x| \, dx.$
7. $\int_0^{4} |x^2 - 4x + 3| \, dx.$

Problem 2. Find $\lim_{x\to\infty} \frac{1}{\sqrt{x}} \int_1^x \frac{dt}{\sqrt{t}}.$

Problem 3. Find the following derivatives.

1. $\frac{d}{dx} \int_0^{\sqrt{x}} \cos t \, dt.$
2. $\frac{d}{dx} \int_1^{\sin x} 3t^2 \, dt.$
3. $\frac{d}{dx} \int_0^{\tan x} \sec^2 t \, dt.$
4. $\frac{d}{dx} \int_0^{\sqrt{x}} \left(t^4 + \frac{3}{\sqrt{1-t^2}}\right) dt.$
5. $\frac{d}{dx} \int_2^{x^2} \sin(t^3) \, dt.$
6. $\frac{d}{dx} \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}} \, dt,\ |x| < \frac{\pi}{2}.$
7. $\frac{d}{dx} \int_0^{\tan x} \frac{1}{1+t^2} \, dt.$

Problem 4. Verify by differentiation that the formula is correct.

1. $\int \frac{1}{x^2\sqrt{1+x^2}} \, dx = -\frac{\sqrt{1+x^2}}{x} + C.$
2. $\int \tan^2 x \, dx = \tan x - x + C.$
3. $\int x\sqrt{a+bx} \, dx = \frac{2}{15b^2}(3bx - 2a)(a+bx)^{\frac{3}{2}} + C.$

Problem 5. Find an anti-derivative of the function $f(x) = |x|$.

Problem 6. Let $G(x) = \int_0^x \left[s \int_0^s f(t) \, dt \right] ds$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous. Find

1. $G(0)$. 2. $G'(0)$. 3. $G''(0)$. 4. $G''(x)$.

Problem 7. Suppose that f has a positive derivative for all values of x (that is, $f'(x) > 0$ for all $x \in \mathbb{R}$) and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t) \, dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of $\frac{dg}{dx}$ crosses the x-axis at $x = 1$.

Problem 8. For each continuous function $f : [0, 1] \to \mathbb{R}$, let

$$I(f) = \int_0^1 x^2 f(x)\,dx \qquad \text{and} \qquad J(f) = \int_0^1 x\,f(x)^2\,dx\,.$$

Find the maximum value of $I(f) - J(f)$ over all such functions f.