Exercise Problem Sets 8

Nov. 05. 2023

Problem 1. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b \cos x\, dx$ by the following steps.

(a) Partition $[a, b]$ into n sub-intervals with equal length. Write down the Riemann sum using the right end-point rule.

(b) Prove that

$$\sum_{i=1}^{n} \cos(a + id) = \frac{\sin\left[a + \left(n + \frac{1}{2}\right)d\right] - \sin\left(a + \frac{d}{2}\right)}{2\sin\frac{d}{2}}\,. \qquad (\star)$$

Hint: Use the sum and difference formula $\sin(\vartheta + \varphi) - \sin(\vartheta - \varphi) = 2\sin\vartheta\cos\varphi$.

(c) Use $(\star)$ to simplify the Riemann sum in (a), and find the limit of the Riemann sum as n approaches infinity. Show that

$$\int_a^b \cos x\, dx = \sin b - \sin a\,.$$

Problem 2. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b x^N\, dx$, where N is a non-negative integer, by the following steps.

(a) Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a regular partition of $[a, b]$. Show that the Riemann sum using the right end-point rule is given by

$$I_n = \sum_{k=0}^{N} \left[C_k^N a^{N-k}(b-a)^{k+1}\left(\frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k\right)\right],$$

where $C_k^N = \dfrac{N!}{k!(N-k)!}$.

(b) Show that

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1}(n+1)^{k+1} - \frac{1}{k+1}\left[C_{k-1}^{k+1}\sum_{i=1}^{n} i^{k-1} + \cdots + C_1^{k+1}\sum_{i=1}^{n} i + (n+1)\right]. \qquad (\star\star)$$

Hint: Expand $(j+1)^k$ for $j = 0, 1, 2, \cdots, n$ by the binomial expansion formula, and sum over j to obtain the equality above.

(c) Use $(\star\star)$ to show that $\displaystyle\lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{i=1}^{n} i^k = \frac{1}{k+1}$ for each $k \in \mathbb{N}$.

(d) Use the limit in (c) to find the limit of the Riemann sum in (a) by passing to the limit as n approaches infinity. Simplify the result to show that

$$\int_a^b x^N\, dx = \frac{b^{N+1} - a^{N+1}}{N+1}\,.$$

Hint: (c) By induction!

Problem 3. Use the limit of Riemann sums to compute the integral $\int_0^{\pi} x \cos x \, dx$.

Problem 4. Let $a > 0$ and $b > 1$. Compute $\int_1^b \log_a x \, dx$ by the following steps.

(a) Partition $[1, a]$ into n sub-intervals by $x_i = r^i$, where $1 \leqslant i \leqslant n$ and $r = b^{\frac{1}{n}}$. Show that the Riemann sum given by the right end-point rule is

$$(r-1)\log_a r \sum_{i=1}^{n} i r^{i-1} \,. \tag{$\diamond$}$$

(b) Use the fact that $\frac{d}{dr} r^i = i r^{i-1}$ to find the sum of $i r^{i-1}$ and show that

$$\sum_{i=1}^{n} i r^{i-1} = \frac{n r^{n+1} - (n+1) r^n + 1}{(r-1)^2} \,. \tag{$\diamond\diamond$}$$

(c) Use ($\diamond$) and ($\diamond\diamond$) to simplify the Riemann sum given by the right end-point rule and show that the Riemann sum is

$$\frac{nbr - nb - b + 1}{n(r-1)} \log_a b = \left[b - \frac{b-1}{n(r-1)} \right] \log_a b \,.$$

(d) Assuming that you know $\left.\frac{d}{dx}\right|_{x=0} b^x = A(a) \log_a b$ for some constant $A > 0$ depending on a, show that

$$\int_1^b \log_a x \, dx = b \log_a b - \frac{b-1}{A(a)} \,.$$