Exercise Problem Sets 7

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Problem 1. 1. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions and $f'(x) = g'(x)$. Show that there exists a constant C such that $f(x) = g(x) + C$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying that $f'(x) = 3x^2 + 4\cos x$ and $f(0) = 0$. Find $f(x)$.

Problem 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that f has only one critical point $c \in (a, b)$.

1. Show that if $f(c)$ is a local extremum of f, then $f(c)$ is an absolute extremum of f.

2. Show that if $f(c)$ is the absolute minimum of f, then $f(x) > f(c)$ for all $x \in [a, b]$ and $x \neq c$. Similarly, show that if $f(c)$ is the absolute maximum of f, then $f(x) < f(c)$ for all $x \in [a, b]$ and $x \neq c$.

Proof. 1. W.L.O.G. we can assume that $f(c)$ is a local minimum of f. By the definition of local extremum, there exists $\delta > 0$ such that

$$f(x) \geqslant f(c) \qquad \forall\, x \in (c - \delta, c + \delta) \subseteq [a, b]\,.$$

Since c is the only critical point of f, there exist $x_1 \in (c - \delta, c)$ and $x_2 \in (c, c + \delta)$ such that $f(x_1) > f(c)$ and $f(x_2) > f(c)$ (for otherwise f is constant in an interval which contradicts to the fact that c is the only critical point of f). This shows that $f(c)$ cannot be the absolute maximum of f.

Suppose the contrary that $f(c)$ is not the absolute minimum of f. Since f is continuous on $[a, b]$, the Extreme Value Theorem and Fermat's Theorem imply that the absolute minimum of f occurs at the end-point.

(a) If $f(a)$ is the absolute minimum of f (with $f(a) < f(c)$), the continuity of f on $[a, c]$ implies that f attains its absolute maximum on $[a, c]$ at some point x_0. It is clear that $x_0 \neq a$. Moreover, since $f(x_1) > f(c)$, $x_0 \neq c$; thus $x_0 \in (a, c)$. By Fermat's Theorem, x_0 is a critical point of f, a contradiction.

(b) Similarly, that $f(b)$ is the absolute minimum of f (with $f(b) < f(c)$) also leads to a contradiction.

Therefore, $f(c)$ has to be the absolute minimum of f.

2. Note that since f has only one critical point c, then f is differentiable on (a, b) except possibly at c. Suppose that there exists another point $d \in [a, b]$, $d \neq c$, such that $f(d) = f(c)$. If $d \in (a, c)$, Rolle's Theorem implies that there exists some point $x_0 \in (d, c)$ such that $f'(x_0) = 0$ which implies that c is not the only critical point, a contradiction. Similarly, that $d \in (c, b)$ also leads to the existence of another critical point in (c, d) which is again a contradiction. ◻

Problem 3. Let I, J be intervals, $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be increasing functions. Show that if J contains the range of g, then $f \circ g$ is increasing on I.

Problem 4. 1. If the function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\frac{2\sqrt{3}}{9}$ at $x = \frac{1}{\sqrt{3}}$, what are the values of a and b?

2. Which of the tangent lines to the curve in part (1) has the smallest slope?

Problem 5. A number a is called a fixed point of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Problem 6. Suppose f is an odd function (that is, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$) and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = \frac{f(b)}{b}$.

Problem 7. Show that $2\sqrt{x} > 3 - \frac{1}{x}$ for all $x > 1$.

Problem 8. Show that $\sqrt{b} - \sqrt{a} < \frac{b-a}{2\sqrt{a}}$ for all $0 < a < b$.

Problem 9. Show that for all (rational numbers) $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$ac + bd \leqslant (a^p + b^p)^{\frac{1}{p}} (c^q + d^q)^{\frac{1}{q}} \qquad \forall\, a, b, c, d > 0\,.$$

Hint: Let $x = \frac{a}{b}$ and $y = \frac{d}{c}$.

Proof. Let $x = \frac{a}{b}$ and $y = \frac{d}{c}$, then the desired inequality is equivalent to that

$$x + y \leqslant (x^p + 1)^{\frac{1}{p}} (y^q + 1)^{\frac{1}{q}} \qquad \forall\, x, y > 0\,.$$

Therefore, it suffices to show the inequality above.

Let $y > 0$ be given. Define

$$f(x) = (x^p + 1)^{\frac{1}{p}} (y^q + 1)^{\frac{1}{q}} - x - y\,.$$

Then

$$f'(x) = \frac{1}{p}(x^p + 1)^{\frac{1-p}{p}} (y^q + 1)^{\frac{1}{q}} \cdot p x^{p-1} - 1 = (1 + x^{-p})^{-\frac{1}{q}} (y^q + 1)^{\frac{1}{q}} - 1\,;$$

thus there is only one critical point of f which is $c = y^{-\frac{q}{p}}$. Now, since

$$f''(c) = \frac{p}{q}\Big(1 + c^{-p}\Big)^{-\frac{1}{q}-1} (y^q + 1)^{\frac{1}{q}} c^{-(p+1)} > 0\,,$$

f attains its local minimum at c. Moreover, since f has only one critical point, f must attain its global minimum at c; thus

$$f(x) \geqslant f(c) \qquad \forall\, x > 0\,.$$

The desired inequality is established by the fact that

$$f(c) = (y^{-q} + 1)^{\frac{1}{p}} (y^q + 1)^{\frac{1}{q}} - y^{-\frac{q}{p}} - y = y \cdot (y^{-q} + 1) - y^{1-q} - y = 0\,.$$

□

Problem 10. Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \leqslant \sin x \leqslant x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} \qquad \forall\, x \geqslant 0\,,$$
$$1 - \frac{x^2}{2!} + \cdots + \frac{x^{4k}}{(4k)!} - \frac{x^{4k+2}}{(4k+2)!} \leqslant \cos x \leqslant 1 - \frac{x^2}{2} + \cdots + \frac{x^{4k}}{(4k)!} \qquad \forall\, x \geqslant 0\,.$$

Problem 11. (不要用交叉相乘) Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$1 - x + x^2 - x^3 + \cdots + x^{2k} - x^{2k+1} \leqslant \frac{1}{1+x} \leqslant 1 - x + x^2 - x^3 + \cdots + x^{2k} \qquad \forall\, x \geqslant 0\,.$$

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying that $f'(x) = f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$.

1. (不要試著找出 f 而是直接用 f 的性質) Show that f is increasing on $\mathbb{R}$.
2. Show that if $k \in \mathbb{N} \cup \{0\}$, then $f(x) \geqslant 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for all $x \geqslant 0$.
3. Show that if $k \in \mathbb{N} \cup \{0\}$, then

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} + \frac{x^{2k+1}}{(2k+1)!} \leqslant f(x) \leqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} \qquad \forall\, x \leqslant 0\,.$$

Hint: 1. Show that f^2 is increasing on $\mathbb{R}$ and argue that f is also increasing on $\mathbb{R}$.

Proof. 1. Since f is differentiable on $\mathbb{R}$, f is continuous on $\mathbb{R}$. By the fact $f(0) = 1$, there exists a interval $[a, b]$, where $a < 0$ and $b > 0$, such that $f > 0$ on $[a, b]$. Since $f'(x) = f(x)$ for all $x \in \mathbb{R}$, we must have $f'(x) > 0$ on $[a, b]$; thus f is increasing on $[a, b]$.

Suppose the contrary that f is not increasing on $[0, \infty)$. Then there exists $c > 0$ such that $f(c) = f'(c) < 0$. Since f is continuous on $[0, c]$, f, restricted to the interval $[0, c]$, attains its maximum at some $x_0 \in [0, c]$. If $x_0 \in (0, c)$, by Fermat's Theorem $f'(x_0) = 0$ which further implies that $f(x_0) = 0$, a contradiction since $f(0) = 1 > f(x_0)$. Therefore, x_0 must be 0 or c. However, f is strictly increasing on $[0, b]$, so $f(0)$ cannot be the maximum of f on $[0, c]$. On the other hand, $f(x_0) < 0 < f(0)$, so $f(x_0)$ cannot be the maximum of f on $[0, c]$. These contradictions lead to the fact that f is increasing on $[0, \infty)$.

Similarly, suppose the contrary that f is not increasing on $(-\infty, 0]$. Then there exists $c < 0$ such that $f(c) = f'(c) < 0$. Since f is continuous, there exists some interval $[c, c+\delta] \subseteq [c, 0]$ such that $f < 0$ on $[c, c+\delta]$. Therefore, f is strictly decreasing on $[c, c+\delta]$. Now, by the continuity of f on $[c, 0]$, f, restricted to the interval $[c, 0]$, attains its minimum at $x_0 \in [c, 0]$. Again, x_0 cannot be 0 since f is increasing on $[a, 0]$, while x_0 cannot be c since f is strictly decreasing on $[c, c+\delta]$. Therefore, $x_0 \in (c, 0)$. Then Fermat's Theorem implies that $f'(x_0) = 0$ which implies that $f(x_0) = 0$ is the minimum of f on $[c, 0]$, a contradiction. Therefore, f is increasing on $(-\infty, 0]$. Combining with the fact that f is increasing on $[0, \infty)$, we find that f is increasing on $\mathbb{R}$.

2. First from the previous step we find that $f(x) \geqslant 1$ for all $x \geqslant 0$. Therefore, the desired inequality holds for the case $k = 0$.

   Assume that the desired inequality holds for the case $k = n$. Define a function $g : [0, \infty) \to \mathbb{R}$ by
$$g(x) = f(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{n+1}}{(n+1)!}\,.$$
   Then
$$g'(x) = f'(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} = f(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}$$
   which, by the assumption that the desired inequality holds for $k = n$, implies that $g'(x) \geqslant 0$. Therefore, $g(x) \geqslant g(0) = 0$ for all $x \geqslant 0$.

3. First from the previous step we find that $f(x) \leqslant 1$ for all $x \leqslant 0$. On the other hand, since
$$\frac{d}{dx}\big[f(x) - 1 - x\big] = f'(x) - 1 = f(x) - 1 \leqslant 0\,,$$
   we find that the function $y = f(x)-1-x$ is decreasing. Therefore, $f(x)-1-x \geqslant f(0)-1-0 = 0$ for all $x \leqslant 0$. This shows that the desired inequality holds for the case $k = 0$.

   Assume that the desired inequality holds for the case $k = n$.

   (a) Define $h_1 : (-\infty, 0] \to \mathbb{R}$ by
$$h_1(x) = f(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{2(n+1)}}{[2(n+1)]!}\,.$$
   Then
$$h_1'(x) = f'(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{2n+1}}{(2n+1)!} = f(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{2n+1}}{(2n+1)!}$$
   which, by the assumption that the desired inequality holds for the case $k = n$, implies that $h_1'(x) \geqslant 0$ for all $x \leqslant 0$. Therefore, h_1 is increasing on $(-\infty, 0]$; thus $h_1(x) \leqslant h_1(0) = 0$ for all $x \leqslant 0$. This implies that
$$f(x) \leqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2(n+1)}}{[2(n+1)]!} \qquad \forall\, x \leqslant 0\,.$$

   (b) Define $h_2 : (-\infty, 0] \to \mathbb{R}$ by
$$h_2(x) = f(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{2(n+1)+1}}{[2(n+1)+1]!}\,.$$
   Then
$$h_2'(x) = f'(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{2n+2}}{(2n+2)!} = f(x) - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^{2(n+1)}}{[2(n+1)]!}$$
   and the (a) implies that $h_2'(x) \leqslant 0$ for all $x \leqslant 0$. Therefore, h_2 is decreasing on $(-\infty, 0]$; thus $h_2(x) \geqslant h_2(0) = 0$ for all $x \leqslant 0$. This implies that
$$f(x) \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2(n+1)+1}}{[2(n+1)+1]!} \qquad \forall\, x \leqslant 0\,.$$

Combining (a) and (b), we find that the desired inequality holds for the case $k = n+1$. By induction, we find that the desired inequality holds for all $k \in \mathbb{N} \cup \{0\}$. □

Problem 13. 1. The function

$$f(x) = \begin{cases} 0 & \text{if } x = 0\,, \\ 1 - x & \text{if } 0 < x \leqslant 1 \end{cases}$$

is differentiable on $(0, 1)$ and satisfies $f(0) = f(1)$. However, its derivative is never zero on $(0, 1)$. Does this contradict Rolle's Theorem? Explain.

2. Can you find a function f such that $f(-2) = -2$, $f(2) = 6$, and $f'(x) < 1$ for all x? Why or why not?

Problem 14. Find the minimum value of

$$\left|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x\right|$$

for real numbers x.
Hint: Let $t = \sin x + \cos x$.

Solution. Let $t = \sin x + \cos x$. Then $t^2 = 1 + 2\sin x \cos x$; thus $\sin x \cos x = \dfrac{t^2 - 1}{2}$. Therefore,

$$\begin{aligned} &\sin x + \cos x + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x} \\ &= \sin x + \cos x + \frac{\sin^2 x + \cos^2 x + \sin x + \cos x}{\sin x \cos x} \\ &= \sin x + \cos x + \frac{1 + \sin x + \cos x}{\sin x \cos x} \\ &= t + \frac{2(1+t)}{t^2 - 1} = t + \frac{2}{t-1} =: f(t)\,. \end{aligned}$$

Define $f(t) = t + \dfrac{2}{t-1}$. Since $-\sqrt{2} \leqslant t \leqslant \sqrt{2}$, we need to find $\min\limits_{t \in [-\sqrt{2}, \sqrt{2}]} \left|f(t)\right|$.

Since $f'(t) = 1 - \dfrac{2}{(t-1)^2}$, we find that $c = 1 - \sqrt{2}$ is the only critical point of f in $\left[-\sqrt{2}, \sqrt{2}\right]$. Finally, since

$$f(1-\sqrt{2}) = 1 - 2\sqrt{2}\,,\ f(-\sqrt{2}) = -\sqrt{2} + \frac{2}{-\sqrt{2}-1} = 2 - 3\sqrt{2}\,,\ f(\sqrt{2}) = \sqrt{2} + \frac{2}{\sqrt{2}-1} = 3\sqrt{2} + 2\,,$$

we find that $\min\limits_{t \in [-\sqrt{2}, \sqrt{2}]} \left|f(t)\right| = 2\sqrt{2} - 1$. □

Problem 15. Let $f, g : (a, b) \to \mathbb{R}$ be twice differentiable functions such that $f''(x) \neq 0$ and $g''(x) \neq 0$ for all $x \in (a, b)$. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).

Problem 16. For what values of a and b is $(2, 2.5)$ an inflection point of the curve $x^2 + ax + by = 0$? What additional inflection points does the curve have?