Exercise Problem Sets 6

Oct. 20. 2023

Problem 1. Find the derivative of the following functions:

1. $y = \cos\sqrt{\sin(\tan(\pi x))}$.
2. $y = \left[x + (x + \sin^2 x)^3\right]^4$.

Problem 2. Let $k \in \mathbb{N}$. Suppose that $\dfrac{d^n}{dx^n}\dfrac{1}{x^k - 1} = \dfrac{p_n(x)}{(x^k - 1)^{n+1}}$ for some polynomial p_n. Find $p_n(1)$ and the degree of p_n.

Problem 3. Let $f_1, f_2, \cdots, f_n : \mathbb{R} \to \mathbb{R}$ be differentiable functions (that is, f_j is differentiable on $\mathbb{R}$ for all $1 \leqslant j \leqslant n$), and

$$h(x) = (f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)(x)\,.$$

Show that

$$h'(x) = f'_n\big(g_{n-1}(x)\big) \cdot f'_{n-2}\big(g_{n-2}(x)\big) \cdot \cdots \cdot f'_2\big(g_1(x)\big) \cdot f'_1(x)\,.$$

where $g_k = f_k \circ f_{k-1} \circ \cdots \circ f_2 \circ f_1$.

Hint: Prove by induction.

Problem 4. 1. Let $r \in \mathbb{Q}$, and $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^r$. Find the derivative of f.

2. Find the derivatives of $y = x^{\frac{1}{4}}$ and $y = x^{\frac{3}{4}}$ by the fact that $x^{\frac{1}{4}} = \sqrt{\sqrt{x}}$ and $x^{\frac{3}{4}} = \sqrt{x\sqrt{x}}$.

3. Let $g : (a, b) \to \mathbb{R}$ be differentiable. Find the derivative of $y = \big|g(x)\big|$.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy $f\big(\dfrac{x^2 - 1}{x^2 + 1}\big) = x$ for all $x > 0$. Find $f'(0)$.

Problem 6. Note that in class we have introduced two new functions "arcsin" and "arccos" whose graphs are (the blue and green) part of the curve consisting of points (x, y) satisfying $\sin y = x$ and $\cos y = x$, respectively, given below

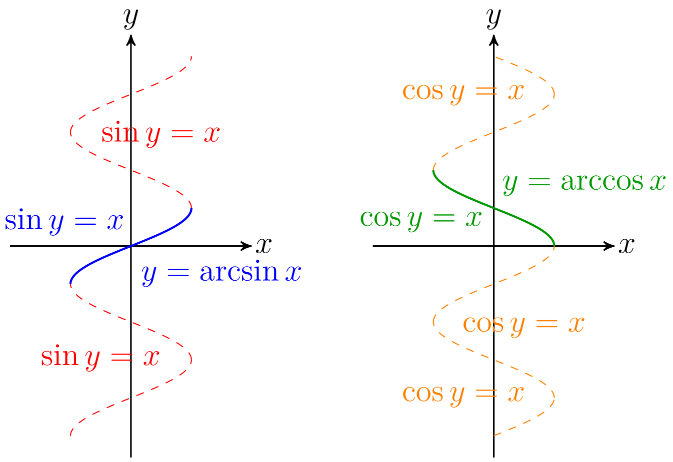

Figure 1: The graph of functions $y = \arcsin x$ and $y = \arccos x$

1. Find the domain and the range of the two functions arcsin and arccos.

2. Show that $\sin(\arcsin x) = x$ for all x in the domain of arcsin and $\cos(\arccos x) = x$ whenever x in the domain of arccos.

3. Is it true that $\arcsin(\sin x) = x$ or $\arccos(\cos x) = x$?

4. Find $\sin(\arccos x)$ and $\cos(\arcsin x)$.

5. Show that $\left.\frac{d}{dx}\right|_{x=c}\big(\arcsin x + \arccos x\big) = 0$ for all c in both domains.

6. Find $\frac{d}{dx}\arcsin\frac{1}{x}$ and $\frac{d}{dx}(\arccos x)^2$.

Problem 7. The function arctan is defined similarly to functions arcsin and arccos: consider the collection of all points (x, y) satisfying $\tan y = x$ (see the figure below), and the blue part is the graph of a function called "arctan".

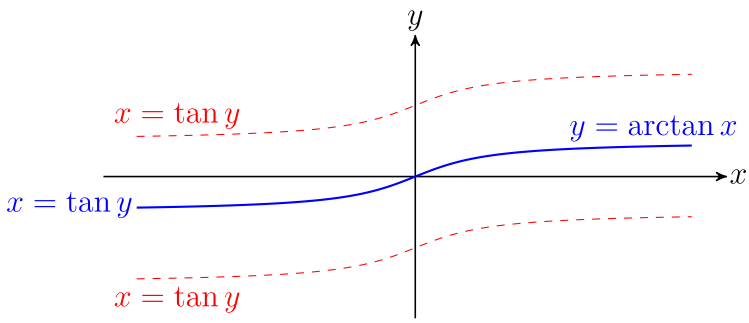

Figure 2: The graph of function $y = \arctan x$

1. Find the domain and the range of the function arctan.

2. Show that $\tan(\arctan x) = x$ for all x in the domain of arctan.

3. Is is true that $\arctan(\tan x) = x$ for all x in the domain of tan?

4. Find $\frac{d}{dx}\arctan x$.

Problem 8. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $\sin(x + y) = y^2 \cos x$.

Problem 9. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$ intersects the curve at what other point?

Problem 10. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y} = \sqrt{c}$ is equal to c.

Problem 11. The Bessel function of order 0, denoted by $y = J_0(x)$, satisfies the differential equation

$$xy'' + y' + xy = 0$$

for all values of x and its value at 0 is $J_0(0) = 1$.

1. Find $J_0'(0)$.

2. Use implicit differentiation to find $J_0''(0)$.