Exercise Problem Sets 14

Dec. 22. 2023

Problem 1. Find the following indefinite integrals.

1. $\displaystyle\int x\csc x\cot x\,dx$
2. $\displaystyle\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx$
3. $\displaystyle\int x\sin^2 x\,dx$
4. $\displaystyle\int \exp(\sqrt[3]{x})\,dx$
5. $\displaystyle\int x\arcsin x\,dx$
6. $\displaystyle\int x\arctan x\,dx$
7. $\displaystyle\int x^2\arctan x\,dx$
8. $\displaystyle\int \ln(x^2-1)\,dx$
9. $\displaystyle\int \sin\sqrt{ax}\,dx$
10. $\displaystyle\int x\tan^2 x\,dx$
11. $\displaystyle\int x^5 e^{-x^3}\,dx$
12. $\displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$
13. $\displaystyle\int \sqrt{x}e^{\sqrt{x}}\,dx$
14. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$
15. $\displaystyle\int \frac{\ln(x+1)}{x^2}\,dx$
16. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-2}}$
17. $\displaystyle\int \sqrt{\tan x}\,dx$
18. $\displaystyle\int x\sin^2 x\cos x\,dx$

Problem 2. The function $y=e^{x^2}$ and $y=x^2e^{x^2}$ don't have elementary anti-derivatives, but $y=(2x^2+1)e^{x^2}$ does. Find the indefinite integral $\displaystyle\int (2x^2+1)e^{x^2}\,dx$.

Solution. Let $u=x$ and $v=e^{x^2}$ (so that $dv=2xe^{x^2}\,dx$). Then

$$\int 2x^2e^{x^2}\,dx = xe^{x^2} - \int ex^2\,dx\,.$$

Therefore, $\displaystyle\int (2x^2+1)e^{x^2}\,dx = \int 2x^2e^{x^2}\,dx + \int e^{x^2}\,dx = xe^{x^2}+C.$ □

Problem 3. Obtain a recursive formula for $\displaystyle\int x^p(ax^n+b)^q\,dx$ and use this relation to find the indefinite integral $\displaystyle\int x^3(x^7+1)^4\,dx$.

Solution. Let $u=(ax^n+b)^q$ and $v=\dfrac{x^{p+1}}{p+1}$ (so that $dv=x^p dx$). Then integration-by-parts implies that

$$\begin{aligned}\int x^p(ax^n+b)^q\,dx &= \frac{x^{p+1}(ax^n+b)^q}{p+1} - \int \frac{x^{p+1}}{p+1}\frac{d}{dx}\big[(ax^n+b)^q\big]\,dx\\ &= \frac{x^{p+1}(ax^n+b)^q}{p+1} - \frac{anq}{p+1}\int x^{p+n}(ax^n+b)^{q-1}\,dx\\ &= \frac{x^{p+1}(ax^n+b)^q}{p+1} - \frac{nq}{p+1}\int x^p(ax^n+b)^q\,dx + \frac{bnq}{p+1}\int x^p(ax^n+b)^{q-1}\,dx\,;\end{aligned}$$

thus

$$\int x^p(ax^n+b)^q\,dx = \frac{x^{p+1}(ax^n+b)^q}{nq+p+1} + \frac{bnq}{nq+p+1}\int x^p(ax^n+b)^{q-1}\,dx\,. \tag{0.1}$$

Therefore, applying (0.1) successively,

$$\begin{aligned}
\int x^3(x^7+1)^4\,dx &= \frac{x^4(x^7+1)^4}{32} + \frac{28}{32}\int x^3(x^7+1)^3\,dx \\
&= \frac{x^4(x^7+1)^4}{32} + \frac{7}{8}\Big[\frac{x^4(x^7+1)^3}{25} + \frac{21}{25}\int x^3(x^7+1)^2\,dx\Big] \\
&= \frac{x^4(x^7+1)^4}{32} + \frac{7x^4(x^7+1)^3}{200} + \frac{147}{200}\Big[\frac{x^4(x^7+1)^2}{18} + \frac{14}{18}\int x^3(x^7+1)\,dx\Big] \\
&= \frac{x^4(x^7+1)^4}{32} + \frac{7x^4(x^7+1)^3}{200} + \frac{147x^4(x^7+1)^2}{3600} + \frac{1029}{1800}\Big[\frac{x^4(x^7+1)}{11} + \frac{7}{11}\int x^3\,dx\Big] \\
&= \frac{x^4(x^7+1)^4}{32} + \frac{7x^4(x^7+1)^3}{200} + \frac{147x^4(x^7+1)^2}{3600} + \frac{1029x^4(x^7+1)}{19800} + \frac{7203}{79200}x^4 + C\,. \qquad \square
\end{aligned}$$

Problem 4. Obtain a recursive formula for $\int x^m(\ln x)^n\,dx$ and use this relation to find the indefinite integral $\int x^4(\ln x)^3\,dx$.

Solution. Let $u = (\ln x)^n$ and $v = \dfrac{x^{m+1}}{m+1}$ (so that $dv = x^m dx$). Then integration-by-parts implies that

$$\int x^m(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \int \frac{x^{m+1}}{m+1}\frac{d}{dx}(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx\,.$$

Therefore,

$$\begin{aligned}
\int x^4(\ln x)^3\,dx &= \frac{x^5(\ln x)^3}{5} - \frac{3}{5}\int x^4(\ln x)^2\,dx = \frac{x^5(\ln x)^3}{5} - \frac{3}{5}\Big[\frac{x^5(\ln x)^2}{5} - \frac{2}{5}\int x^4\ln x\,dx\Big] \\
&= \frac{x^5(\ln x)^3}{5} - \frac{3x^5(\ln x)^2}{25} + \frac{6}{25}\Big[\frac{x^5\ln x}{5} - \frac{1}{5}\int x^4\,dx\Big] \\
&= \frac{x^5(\ln x)^3}{5} - \frac{3x^5(\ln x)^2}{25} + \frac{6x^5\ln x}{125} - \frac{6}{625}x^5 + C\,. \qquad \square
\end{aligned}$$

Problem 5. Find the following integrals.

1. $\displaystyle\int \sin^2 x\cos^4 x\,dx.$ 2. $\displaystyle\int x\sin^2 x\cos^4 x\,dx.$ 3. $\displaystyle\int x^2\sin^2 x\cos^4 x\,dx.$

Proof. 1. By the half angle formula and the triple angle formula, we find that

$$\begin{aligned}
\sin^2 x\cos^4 x &= \Big[\frac{1-\cos(2x)}{2}\Big]\Big[\frac{1+\cos(2x)}{2}\Big]^2 = \frac{1}{8}\big[1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big] \\
&= \frac{1}{8}\Big[1+\cos(2x) - \frac{1+\cos(4x)}{2} - \frac{\cos(6x)+3\cos(2x)}{4}\Big] \\
&= \frac{1}{32}\Big[2+\cos(2x)-2\cos(4x)-\cos(6x)\Big]\,.
\end{aligned}$$

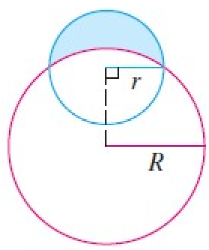
Therefore,

$$\begin{aligned}
\int \sin^2 x\cos^4 x\,dx &= \frac{1}{32}\int\Big[2+\cos(2x)-2\cos(4x)-\cos(6x)\Big]\,dx \\
&= \frac{1}{32}\Big[2x + \frac{\sin(2x)}{2} - \frac{\sin(4x)}{2} - \frac{\sin(6x)}{6}\Big] + C \\
&= \frac{1}{192}\big[12x + 3\sin(2x) - 3\sin(4x) - \sin(6x)\big] + C\,.
\end{aligned}$$

2. Let $u(x) = x$ and $v(x) = \frac{1}{192}\big[12x + 3\sin(2x) - 3\sin(4x) - \sin(6x)\big]$. Then we integrate by parts to obtain

$$\begin{aligned}\int x\sin^2 x\cos^4 x\,dx &= \int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx \\ &= \frac{x}{192}\big[12x + 3\sin(2x) - 3\sin(4x) - \sin(6x)\big] \\ &\quad - \frac{1}{192}\int\big[12x + 3\sin(2x) - 3\sin(4x) - \sin(6x)\big]\,dx \\ &= \frac{x}{192}\big[12x + 3\sin(2x) - 3\sin(4x) - \sin(6x)\big] \\ &\quad - \frac{1}{192}\Big[6x^2 - \frac{3}{2}\cos(2x) + \frac{3}{4}\cos(4x) + \frac{1}{6}\cos(6x)\Big] + C \\ &= \frac{1}{192}\big[6x^2 + 3x\sin(2x) - 3x\sin(4x) - x\sin(6x)\big] \\ &\quad + \frac{1}{192}\Big[\frac{3}{2}\cos(2x) - \frac{3}{4}\cos(4x) - \frac{1}{6}\cos(6x)\Big] + C\,.\end{aligned}$$ □

Problem 6. Find the area of the crescent-shaped region (called a lune) bounded by arcs of circles with radii r and R. (See the figure)



Problem 7. Complete the following.

1. Let $f : [a, b] \to [c, d]$ be a continuously differentiable increasing function, and $f([a, b]) = [c, d]$. Suppose that f has an inverse f^{-1}. Show that
$$\int_a^b f(x)\,dx + \int_c^d f^{-1}(y)\,dy = bf(b) - af(a)\,. \tag{0.2}$$
2. How about if f is decreasing?
3. Use (0.2) to compute $\int_0^1 \arcsin x\,dx$ and $\int_0^1 \arctan x\,dx$.
4. Let F be an anti-derivative of a continuously differentiable function f with inverse f^{-1}. Find an anti-derivative of f^{-1} in terms of f and F.

Proof. 1. Since f is increasing, $c = f(a)$ and $d = f(b)$. The substitution of variable formula then implies that
$$\int_c^d f^{-1}(y)\,dy \overset{(y=f(x))}{=} \int_a^b xf'(x)\,dx\,.$$
Integrating by parts (with $u = x$ and $v = f(x)$),
$$\int_a^b xf'(x)\,dx = xf(x)\Big|_{x=a}^{x=b} - \int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx\,;$$

thus

$$\int_a^b f(x)\,dx + \int_c^d f^{-1}(y)\,dx = bf(b) - af(a)\,.$$

2. Since f is decreasing, $d = f(a)$ and $c = f(b)$. The substitution of variable formula then implies that

$$\int_d^c f^{-1}(y)\,dy \overset{(y=f(x))}{=} \int_a^b xf'(x)\,dx\,.$$

Integrating by parts (with $u = x$ and $v = f(x)$) as in 1, we conclude that

$$\int_a^b xf'(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx\,;$$

thus

$$\int_a^b f(x)\,dx + \int_d^c f^{-1}(y)\,dx = bf(b) - af(a)\,.$$

Therefore, 1 and 2 together imply that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dx = bf(b) - af(a)\,.$$

3. Let $f(x) = \sin x$ and $a = 0$, $b = \dfrac{\pi}{2}$ in 1, we have

$$\int_0^{\frac{\pi}{2}} \sin x\,dx + \int_0^1 \arcsin y\,dy = \frac{\pi}{2}\sin\frac{\pi}{2} - 0\cdot\sin 0 = \frac{\pi}{2}\,.$$

Therefore, $\displaystyle\int_0^1 \arcsin x\,dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{2} + \Big(\cos x\Big|_{x=0}^{x=\frac{\pi}{2}}\Big) = \frac{\pi}{2} + 1.$

Let $f(x) = \tan x$ and $a = 0$, $b = \dfrac{\pi}{4}$ in 1, we have

$$\int_0^{\frac{\pi}{4}} \tan x\,dx + \int_0^1 \arctan y\,dy = \frac{\pi}{4}\tan\frac{\pi}{4} - 0\cdot\tan 0 = \frac{\pi}{4}\,.$$

Therefore, $\displaystyle\int_0^1 \arctan x\,dx = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{\pi}{4} - \Big(\ln|\sec x|\Big|_{x=0}^{x=\frac{\pi}{4}}\Big) = \frac{\pi}{4} - \ln\sqrt{2}.$ □

Problem 8. For $n \in \mathbb{N} \cup \{0\}$, the Legendre polynomial of degree n, denoted by P_n, is defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\,.$$

1. Show that $\displaystyle\int_{-1}^1 P_n(x)P_m(x)\,dx = 0$ if $m \neq n$.

2. Show that $\displaystyle\int_{-1}^1 P_n(x)^2\,dx = \frac{2}{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

3. Show that $\displaystyle\int_{-1}^1 x^m P_n(x)\,dx = 0$ if $m < n$.

4. Evaluate $\displaystyle\int_{-1}^1 x^n P_n(x)\,dx$.

Proof. Before proceeding, we note that if $0 \leqslant k < n$, then

$$\frac{d^k}{dx^k}\Big|_{x=\pm 1}(x^2-1)^n = 0 \tag{0.3}$$

since Leibniz's rule, with the help of the identity $\dfrac{d^j}{dx^j}x^n = \dfrac{n!}{(n-j)!}x^{n-j}$, implies that

$$\begin{aligned}\frac{d^k}{dx^k}(x^2-1)^n &= \sum_{\ell=0}^{k} C^k_\ell \Big(\frac{d^\ell}{dx^\ell}(x-1)^n\Big)\Big(\frac{d^{k-\ell}}{dx^{k-\ell}}(x+1)^n\Big)\\ &= \sum_{\ell=0}^{k} \frac{C^k_\ell (n!)^2}{(n-k+\ell)!(n-\ell)!}(x-1)^{n-\ell}(x+1)^{n-k+\ell}\,.\end{aligned}$$

Let $v_k = \dfrac{d^k}{dx^k}(x^2-1)^n$. Then $v_{k+1} = v_k'$ for all $k \geqslant 0$ and $v_k(\pm 1) = 0$ for all $0 \leqslant k \leqslant n-1$.

Let q be a polynomail. For $0 \leqslant k \leqslant n-1$, integrating by parts leads to that

$$\begin{aligned}\int_{-1}^{1} q(x)v_{k+1}(x)\,dx &= \int_{-1}^{1} q(x)v_k'(x)\,dx = q(x)v_k(x)\Big|_{x=-1}^{x=1} - \int_{-1}^{1} q'(x)v_k(x)\,dx\\ &= -\int_{-1}^{1} q'(x)v_k(x)\,dx\,.\end{aligned}$$

Therefore, if q is a polynomial,

$$\begin{aligned}\int_{-1}^{1} q(x)\frac{d^n}{dx^n}(x^2-1)^n\,dx &= \int_{-1}^{1} q(x)v_n(x)\,dx = -\int_{-1}^{1} q'(x)v_{n-1}(x)\,dx\\ &= (-1)^2\int_{-1}^{1} q''(x)v_{n-2}(x)\,dx = \cdots = (-1)^\ell \int_{-1}^{1} q^{(\ell)}(x)v_{n-\ell}(x)\,dx\\ &= \cdots = (-1)^n\int_{-1}^{1} q^{(m)}(x)v_0(x)\,dx\,.\end{aligned} \tag{0.4}$$

Letting $q(x) = \dfrac{1}{2^m m!}P_m(x)$ in the identity above, we conclude that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{(-1)^m}{2^{n+m}n!m!}\int_{-1}^{1}\Big(\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^m\Big)(x^2-1)^n\,dx\,. \tag{0.5}$$

1. W.L.O.G. we assume that $n > m$. Then (0.5) shows that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$$

since $\dfrac{d^{m+m}}{dx^{m+n}}(x^2-1)^m = 0$ for $n > m$.

2. Let $m = n$ in (0.5), we obtain that

$$\begin{aligned}\int_{-1}^{1} P_n(x)P_n(x)\,dx &= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}\Big(\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\Big)(x^2-1)^n\,dx\\ &= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(2n)!(x^2-1)^n\,dx \overset{(x=\sin t)}{=} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-1)^n\cos^{2n+1}t\,dt\\ &= \frac{(2n)!}{2^{2n-1}(n!)^2}\int_{0}^{\frac{\pi}{2}}\cos^{2n+1}t\,dt\,,\end{aligned}$$

where the last equality follows from that cosine is an even function. By Wallis' formula,

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} t\,dt = \frac{(2^n n!)^2}{(2n+1)!}\,;$$

thus

$$\int_{-1}^{1} P_n(x)P_n(x)\,dx = \frac{(2n)!}{2^{2n-1}(n!)^2}\frac{(2^n n!)^2}{(2n+1)!} = \frac{2}{2n+1}\,.$$

3. Part 3 follows from the identity (0.4) with $q(x) = x^m$.

4. Using the identity (0.4) with $q(x) = x^n$, similar to part 2 we find that

$$\int_{-1}^{1} x^n \frac{d^n}{dx^n}(x^2-1)^n\,dx = (-1)^n \int_{-1}^{1} n!(x^2-1)^n\,dx = 2\cdot n! \int_0^{\frac{\pi}{2}} \cos^{2n+1} t\,dt$$
$$= \frac{2(2^n n!)^2 n!}{(2n+1)!}\,.$$

Therefore,

$$\int_{-1}^{1} x^n P_n(x) = \frac{1}{2^n n!}\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n\,dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}\,.$$

□

Problem 9. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be distinct real numbers, and

$$g(x) = \prod_{k=1}^{n}(x-\alpha_k) \equiv (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)\,.$$

Use the partial fraction expansion to prove Newton's formula

$$\frac{\alpha_1^k}{g'(\alpha_1)} + \frac{\alpha_2^k}{g'(\alpha_2)} + \cdots + \frac{\alpha_n^k}{g'(\alpha_n)} = \begin{cases} 0 & \text{for } k = 0,1,2,\cdots,n-2\,, \\ 1 & \text{for } k = n-1\,. \end{cases}$$

Hint: By partial fraction, for $k < n-1$

$$\frac{x^k}{(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)} = \frac{A_2}{x-\alpha_2} + \frac{A_3}{x-\alpha_3} + \cdots + \frac{A_n}{x-\alpha_n}\,.$$

Show that $A_j = \dfrac{\alpha_j^k(\alpha_j-\alpha_1)}{g'(\alpha_j)}$ and conclude from here. Do the same for the case $k = n-1$.

Solution. 1. If $k < n-1$, then

$$\frac{x^k}{(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)} = \frac{A_2}{x-\alpha_2} + \frac{A_3}{x-\alpha_3} + \cdots + \frac{A_n}{x-\alpha_n}\,,$$

where

$$A_j = \frac{\alpha_j^k}{(\alpha_j-\alpha_2)(\alpha_j-\alpha_3)\cdots(\alpha_j-\alpha_{j-1})(\alpha_j-\alpha_{j+1})\cdots(\alpha_j-\alpha_n)} = \frac{\alpha_j^k(\alpha_j-\alpha_1)}{g'(\alpha_j)}\,.$$

Therefore,

$$\frac{x^k}{(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)} = \sum_{j=2}^{n} \frac{\alpha_j^k(\alpha_j-\alpha_1)}{g'(\alpha_j)(x-\alpha_j)}$$

which, by letting $x = \alpha_1$, further implies that

$$\frac{\alpha_1^k}{g'(\alpha_1)} = \frac{\alpha_1^k}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)\cdots(\alpha_1 - \alpha_n)} = \sum_{j=2}^{n} \frac{\alpha_j^k(\alpha_j - \alpha_1)}{g'(\alpha_j)(\alpha_1 - \alpha_j)} = -\sum_{j=2}^{n} \frac{\alpha_j^k}{g'(\alpha_j)}\,;$$

thus if $k < n-1$,

$$\sum_{j=1}^{n} \frac{\alpha_j^k}{g'(\alpha_j)} = 0\,.$$

2. If $k = n-1$, then

$$\frac{x^{n-1}}{(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)} = 1 + \frac{A_2}{x-\alpha_2} + \frac{A_3}{x-\alpha_3} + \cdots + \frac{A_n}{x-\alpha_n}\,,$$

where

$$A_j = \frac{\alpha_j^{n-1}}{(\alpha_j-\alpha_2)(\alpha_j-\alpha_3)\cdots(\alpha_j-\alpha_{j-1})(\alpha_j-\alpha_{j+1})\cdots(\alpha_j-\alpha_n)} = \frac{\alpha_j^{n-1}(\alpha_j-\alpha_1)}{g'(\alpha_j)}\,.$$

In other words, for $0 \leqslant k \leqslant n-1$,

$$\frac{x^k}{(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)} = \delta_{k(n-1)} + \sum_{j=2}^{n} \frac{\alpha_j^k(\alpha_j-\alpha_1)}{g'(\alpha_j)(x-\alpha_j)}\,,$$

where $\delta_{k(n-1)}$ is the Kronecker delta satisfying $\delta_{k(n-1)} = 1$ if $k = n-1$ and $\delta_{k(n-1)} = 0$ if $0 \leqslant k < n-1$. Repeat the procedure for the case $k < n-1$, we conclude that $\sum\limits_{j=1}^{n} \dfrac{\alpha_j^{n-1}}{g'(\alpha_j)} = 1$. ▫