Calculus 微積分

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Chapter 14

Multiple Integration

14.1 Double Integrals and Volume

Let R be a closed and bounded region in the plane, and $f : R \to \mathbb{R}$ be a non-negative continuous function. We are interested in the volume of the solid in space

$$\mathrm{D} = \big\{(x, y, z) \,\big|\, (x, y) \in R\,, 0 \leqslant z \leqslant f(x, y)\big\}\,.$$

First we assume that $R = [a, b] \times [c, b] = \big\{(x, y) \,\big|\, a \leqslant x \leqslant b, c \leqslant y \leqslant d\big\}$ be a rectangle. Let $\mathcal{P}_x = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and $\mathcal{P}_y = \{c = y_0 < y_1 < \cdots < y_m = d\}$ be partitions of $[a, b]$ and $[c, d]$, respectively, R_{ij} denote the rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, and $\big\{(\alpha_i, \beta_j)\big\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ be a collection of points such that $\alpha_i \in [x_{i-1}, x_i]$ and $\beta_j \in [y_{j-1}, y_j]$. Then as before, we consider an approximation of the volume of D given by

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\alpha_i, \beta_j)(x_i - x_{i-1})(y_j - y_{j-1})\,.$$

Then the limit of the sum above, as $\|\mathcal{P}_x\|$, $\|\mathcal{P}_y\|$ approaches zero, is the volume of D. The collection of rectangles $\mathcal{P} = \{R_{ij}\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ is called a partition of R.

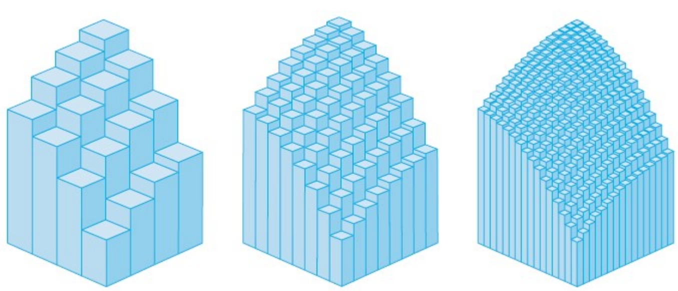


Figure 14.1: The volume of D can be obtained by making $\|\mathcal{P}_x\|, \|P_y\| \to 0$.

In general, by relabeling the rectangles as $R_1, R_2, \cdots, R_{nm}$ (thus $\mathcal{P} = \{R_k \,|\, 1 \leqslant k \leqslant nm\}$), and letting $\big\{(\xi_k, \eta_k)\big\}_{k=1}^{nm}$ be a collection of point in R such that $(\xi_k, \eta_k) \in R_k$, we can consider an approximation of the volume of the solid given by

$$\sum_{k=1}^{nm} f(\xi_k, \eta_k) A_k \,,$$

where A_k is the area of the rectangle R_k. The sum above is called a ***Riemann sum of f for partition $\mathcal{P}$***. With $\|\mathcal{P}\|$, called the norm of $\mathcal{P}$, denoting the maximum length of the diagonal of R_k; that is,

$$\|\mathcal{P}\| = \max\big\{\ell_k \,\big|\, \ell_k \text{ is the length of the diagonal of } R_k, 1 \leqslant k \leqslant nm\big\}\,,$$

then the volume of D is the "limit"

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{k=1}^{nm} f(\xi_k, \eta_k) A_k$$

as long as "the limit exists". Similar to the discussion of the limit of Riemann sums in the case of functions of one variable, we can remove the restrictions that f is continuous and non-negative on R and still consider the limit of the Riemann sums. We have the following

Definition 14.1

Let $R = [a, b] \times [c, d]$ be a rectangle in the plane, and $f : R \to \mathbb{R}$ be a function. f is said to be Riemann integrable on R if there exists a real number V such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of R satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(V - \varepsilon, V + \varepsilon)$. Such a number V (is unique if it exists and) is called the ***Riemann integral*** or ***double integral of f on R*** and is denoted by $\displaystyle\iint_R f(x, y)\, dA$ or simply $\displaystyle\int_R f(x, y)\, d(x, y)$.

How about the case that the base R of the solid is not a closed and bounded rectangle? In this case we choose $r > 0$ large enough such that $R \subseteq [-r, r]^2 \equiv [-r, r] \times [-r, r]$ and then for a function $f : R \to \mathbb{R}$, define $\widetilde{f} : [-r, r]^2 \to \mathbb{R}$ by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in R\,, \\ 0 & \text{if } x \notin R\,. \end{cases}$$

We define $\displaystyle\iint_R f(x, y)\, dA$ as $\displaystyle\iint_{[-r,r]^2} \widetilde{f}(x, y)\, dA$ (when the latter double integral exists).

Before proceeding, let us talk about a special class of regions.

Definition 14.2

A region R is said to be have area if the constant function 1 is Riemann integrable on R. If R has area, then the area of R is defined as the integral $\displaystyle\iint\limits_R 1\,dA$.

The following theorem is an analogy of Theorem 4.10.

Theorem 14.3

Let R be a closed and bounded region in the plane, and $f : R \to \mathbb{R}$ be a function. If R has area and f is continuous on R, then f is Riemann integrable on R.

Similar to the properties for integrals of functions of one variable, we have the following

Theorem 14.4: Properties of double integrals

Let R be a closed and bounded region in the plane, $f, g : R \to \mathbb{R}$ be functions that are Riemann integrable on R, and c be a real number.

1. cf is Riemann integrable on R, and
$$\iint\limits_R (cf)(x,y)\,dA = c\iint\limits_R f(x,y)\,dA\,.$$
2. $f \pm g$ are Riemann integrable on R, and
$$\iint\limits_R (f \pm g)(x,y)\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA\,.$$
3. If $f(x,y) \geqslant g(x,y)$ for all $(x,y) \in R$, then
$$\iint\limits_R f(x,y)\,dA \geqslant \iint\limits_R g(x,y)\,dA\,.$$
4. $|f|$ is Riemann integrable, and
$$\Big|\iint\limits_R f(x,y)\,dA\Big| \leqslant \iint\limits_R \big|f(x,y)\big|\,dA\,.$$

Definition 14.5

Two bounded regions R_1 and R_2 in the plane are said to be non-overlapping if $R_1 \cap R_2$ has zero area.

Theorem 14.6

Let R_1 and R_2 be non-overlapping regions in the plane, $R = R_1 \cup R_2$, and $f : R \to \mathbb{R}$ be such that f is Riemann integrable on R_1 and R_2. Then f is Riemann integrable on R and

$$\iint\limits_{R} f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA\,.$$

14.2 The Iterated Integrals and Fubini's Theorem

Let R be a bounded region with area, and $f : R \to \mathbb{R}$ be a non-negative continuous function. As explained in the previous section, the volume of the solid

$$\mathrm{D} = \big\{(x,y,z)\,\big|\,(x,y) \in R, 0 \leqslant z \leqslant f(x,y)\big\}$$

is given by $\displaystyle\iint\limits_{R} f(x,y)\,dA$. We are concerned with computing this double integral in this section.

Recall from Section 7.2 that if D is a solid lies between two planes $x = a$ and $x = b$ $(a < b)$, and the area of the cross section of D taken perpendicular to the x-axis is $A(x)$, then

$$\text{the volume of D} = \int_a^b A(x)\,dx\,.$$

Therefore, if the region R is given by

$$R = \big\{(x,y)\,\big|\,a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\big\}$$

for some continuous functions $g_1, g_2 : [a,b] \to \mathbb{R}$, then the area of the cross section of D taken perpendicular to the x axis is

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

which shows that the volume of D is given by $\int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \Big)\,dx$. Therefore, in this special case we find that

$$\iint\limits_R f(x,y)\,dA = \int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \Big)\,dx\,. \tag{14.2.1}$$

Similarly, recall that if D lies between $y = c$ and $y = d$ ($c < d$), and the area of the cross section of D taken perpendicular to the y-axis is $A(y)$, then

$$\text{the volume of D} = \int_c^d A(y)\,dy\,;$$

thus similar argument shows that

$$\iint\limits_R f(x,y)\,dA = \int_c^d \Big(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \Big)\,dy\,. \tag{14.2.2}$$

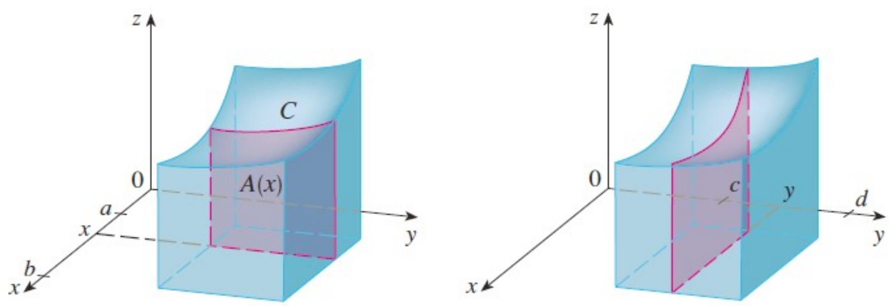


Figure 14.2: Finding the volume of D using the method of cross section

We note that in formulas (14.2.1), we have to compute the integral $\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$ for each fixed $x \in [a,b]$ which gives the area of the cross section $A(x)$, then compute the integral $\int_a^b A(x)\,dx$ to obtain the volume of D. This way of computing double integrals is called ***iterated integrals***, and sometime we omit the parentheses and write it as

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx\,.$$

Similarly, the iterated integral appearing in (14.2.2) can also be written as

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy\,.$$

The evaluation of the double integral $\displaystyle\iint\limits_R f(x,y)\,dA$ can be generalized for a more general class of functions, and it is called the Fubini Theorem.

Theorem 14.7: Fubini's Theorem

Let R be a region in the plane, and $f: R \to \mathbb{R}$ be continuous (but no necessary non-negative).

1. If R is given by $R = \big\{(x,y)\,\big|\,a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\big\}$, then
$$\iint\limits_R f(x,y)\,dA = \int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\Big)\,dx\,.$$
2. If R is given by $R = \big\{(x,y)\,\big|\,c \leqslant y \leqslant d, g_1(x) \leqslant y \leqslant g_2(x)\big\}$, then
$$\iint\limits_R f(x,y)\,dA = \int_c^d \Big(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\Big)\,dy\,.$$

Example 14.8. Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane. By the definition of double integrals, the volume of this solid is given by $\displaystyle\iint\limits_R (4 - x^2 - 2y^2)\,dA$, where R is the region $\big\{(x,y)\,\big|\,x^2 + 2y^2 \leqslant 4\big\}$. Writing R as
$$R = \Big\{(x,y)\,\Big|\,-2 \leqslant x \leqslant 2\,, -\sqrt{\frac{4-x^2}{2}} \leqslant y \leqslant \sqrt{\frac{4-x^2}{2}}\Big\}$$
or
$$R = \big\{(x,y)\,\big|\,-\sqrt{2} \leqslant y \leqslant \sqrt{2}\,, -\sqrt{4-2y^2} \leqslant x \leqslant \sqrt{4-2y^2}\big\}\,,$$
the Fubini Theorem then implies that
$$\begin{aligned}\iint\limits_R (4 - x^2 - 2y^2)\,dA &= \int_{-2}^{2} \Big(\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4 - x^2 - 2y^2)\,dy\Big)\,dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \Big(\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} (4 - x^2 - 2y^2)\,dx\Big)\,dy\,.\end{aligned}$$

1. Integrating in y first then integrating in x: for fixed $x \in [-2, 2]$,

$$\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4-x^2-2y^2)\,dy = \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4-x^2)\,dy - 2\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} y^2\,dy$$
$$= \sqrt{2}(4-x^2)^{\frac{3}{2}} - \frac{4}{3}\Big(\sqrt{\frac{4-x^2}{2}}\Big)^3 = \frac{2\sqrt{2}}{3}(4-x^2)^{\frac{3}{2}}\,.$$

Therefore, by the substitution $x = 2\sin\theta$ (so that $dx = 2\cos\theta\,d\theta$),

$$\iint_R (4-x^2-2y^2)\,dA = \frac{2\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{\frac{3}{2}}\,dx = \frac{2\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta\cdot 2\cos\theta d\theta$$
$$= \frac{32\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta = \frac{64\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$
$$= \frac{64\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\Big(\frac{1+\cos 2\theta}{2}\Big)^2 d\theta$$
$$= \frac{16\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\Big(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\Big)\,d\theta$$
$$= \frac{16\sqrt{2}}{3}\Big[\frac{3}{2}\cdot\frac{\pi}{2}+\sin\Big(2\cdot\frac{\pi}{2}\Big)+\frac{1}{8}\sin\Big(4\cdot\frac{\pi}{2}\Big)\Big] = 4\sqrt{2}\pi\,.$$

2. Integrating in x first then integrating in y: for fixed $y\in[-\sqrt{2},\sqrt{2}]$,

$$\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}(4-x^2-2y^2)\,dx = \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}(4-2y^2)\,dx - \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} x^2\,dx$$
$$= 2(4-2y^2)^{\frac{3}{2}} - \frac{2}{3}(4-2y^2)^{\frac{3}{2}} = \frac{4}{3}(4-2y^2)^{\frac{3}{2}}\,;$$

thus by the substitution of variable $y=\sqrt{2}\sin\theta$ (so that $dy = \sqrt{2}\cos\theta\,d\theta$),

$$\iint_R (4-x^2-2y^2)\,dA = \frac{4}{3}\int_{-\sqrt{2}}^{\sqrt{2}}(4-2y^2)^{\frac{3}{2}}\,dy = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta\cdot\sqrt{2}\cos\theta\,d\theta$$
$$= \frac{32\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta = \frac{64\sqrt{2}}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta = 4\sqrt{2}\pi\,.$$

Example 14.9. Find the volume of the solid region bounded above by the paraboloid $z = 1-x^2-y^2$ and below by the plane $z = 1-y$.

Let R be the region in the plane whose boundary points (x,y) satisfies $1-x^2-y^2 = 1-y$ or equivalently, $x^2+y^2-y=0$. Then the volume of the solid described above is given by

$\displaystyle\iint\limits_R \big[(1-x^2-y^2)-(1-y)\big]\,dA$. Note that the region R is a disk centered at $\big(0,\frac{1}{2}\big)$ with radius $\frac{1}{2}$ and can be written as

$$R=\big\{(x,y)\,\big|\,0\leqslant y\leqslant 1\,,-\sqrt{y-y^2}\leqslant x\leqslant\sqrt{y-y^2}\big\}\,.$$

Therefore,

$$\begin{aligned}\iint\limits_R \big[(1-x^2-y^2)-(1-y)\big]\,dA &= \int_0^1\Big(\int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}}(y-x^2-y^2)\,dx\Big)dy\\ &=\int_0^1\Big(2(y-y^2)^{\frac{3}{2}}-\frac{2}{3}(y-y^2)^{\frac{3}{2}}\Big)\,dy=\frac{4}{3}\int_0^1(y-y^2)^{\frac{3}{2}}\,dy=\frac{4}{3}\int_0^1\Big[\frac{1}{4}-\big(y-\frac{1}{2}\big)^2\Big]^{\frac{3}{2}}\,dy\,.\end{aligned}$$

Making the substitution of variable $y-\frac{1}{2}=\frac{1}{2}\sin\theta$ $\big($so that $dy=\frac{1}{2}\cos\theta\,d\theta\big)$,

$$\iint\limits_R \big[(1-x^2-y^2)-(1-y)\big]\,dA=\frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos^3\theta}{8}\cdot\frac{1}{2}\cos\theta\,d\theta=\frac{1}{6}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta=\frac{\pi}{32}\,.$$

Example 14.10. Find the iterated integral $\displaystyle\int_0^1\Big(\int_y^1 e^{-x^2}\,dx\Big)dy$.

Let $R=\big\{(x,y)\,\big|\,0\leqslant y\leqslant 1\,,y\leqslant x\leqslant 1\big\}$. Since R can also be expressed as $R=\big\{(x,y)\,\big|\,0\leqslant x\leqslant 1\,,0\leqslant y\leqslant x\big\}$, by the Fubini Theorem we find that

$$\begin{aligned}\int_0^1\Big(\int_y^1 e^{-x^2}\,dx\Big)\,dy&=\iint\limits_R e^{-x^2}\,dA=\int_0^1\Big(\int_0^x e^{-x^2}\,dy\Big)dx=\int_0^1 xe^{-x^2}\,dx\\ &=-\frac{1}{2}e^{-x^2}\Big|_{x=0}^{x=1}=\frac{1}{2}\big(1-e^{-1}\big)\,.\end{aligned}$$

14.3 Surface Area

14.3.1 Surface area of graph of functions

Let $R=[a,b]\times[c,d]$ be a rectangle in the plane, and $f:R\to\mathbb{R}$ be a continuously differentiable function. We are interested in the area of the surface

$$S=\big\{(x,y,z)\,\big|\,(x,y)\in R\,,z=f(x,y)\big\}\,.$$

Let $\mathcal{P}=\big\{R_{ij}\,\big|\,1\leqslant i\leqslant n\,,1\leqslant j\leqslant m\big\}$ be a partition of R. Partition each rectangle $R_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$ into two triangles Δ_{ij}^1 and Δ_{ij}^2, where Δ_{ij}^1 has vertices (x_{i-1},y_{j-1}),

(x_i, y_{j-1}), (x_{i-1}, y_j) and Δ^2_{ij} has vertices (x_i, y_j), (x_{i-1}, y_j), (x_i, y_{j-1}). Then intuitively, the area of the surface $f(\Delta^1_{ij})$ can be approximated by the area of the triangle T^1_{ij} with vertices $\big(x_{i-1}, y_{j-1}, f(x_{i-1}, y_{j-1})\big)$, $\big(x_i, y_{j-1}, f(x_i, y_{j-1})\big)$ and $\big(x_i, y_j, f(x_i, y_j)\big)$, while the area of the surface $f(\Delta^2_{ij})$ can be approximated by the area of the triangle T^2_{ij} with vertices $\big(x_i, y_j, f(x_i, y_j)\big)$, $\big(x_{i-1}, y_j, f(x_{i-1}, y_j)\big)$ and $\big(x_i, y_{j-1}, f(x_i, y_{j-1})\big)$. Therefore, the area of the surface $f(R_{ij})$ can be approximated by the sum of area of triangles T^1_{ij} and T^2_{ij}, and the area of the surface S can be approximated by the sum of the area of the triangles T^1_{ij} and T^2_{ij}, where is sum is taken over all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Now we compute the area of the triangles T^1_{ij} and T^2_{ij}. We remark that for a triangle T with vertices P_1, P_2, P_3, letting $\boldsymbol{u} = \overrightarrow{P_1P_2} = P_2 - P_1$ and $\boldsymbol{v} = \overrightarrow{P_1P_3} = P_3 - P_1$, the area of T can be computed by $\dfrac{1}{2}\|\boldsymbol{u} \times \boldsymbol{v}\|$. Therefore, the area of T^1_{ij} is given by

$$\begin{aligned} |T^1_{ij}| = \frac{1}{2}\Big\|\big(x_i - x_{i-1}, 0, f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1})\big) \times \\ \times \big(0, y_j - y_{j-1}, f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1})\big)\Big\| . \end{aligned}$$

By the mean value theorem, there exist $\xi^*_i \in (x_{i-1}, x_i)$ and $\eta^*_j \in (y_{j-1}, y_j)$ such that

$$\begin{aligned} f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1}) &= f_x(\xi^*_i, y_{j-1})(x_i - x_{i-1}) \, , \\ f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1}) &= f_y(x_{i-1}, \eta^*_j)(y_j - y_{j-1}) \, ; \end{aligned}$$

thus we obtain that

$$\begin{aligned} |T^1_{ij}| &= \frac{1}{2}\,\big\|\big(1, 0, f_x(\xi^*_i, y_{j-1})\big) \times \big(0, 1, f_y(x_{i-1}, \eta^*_j)\big)\big\| \\ &= \frac{1}{2}\,\big\|\big(-f_x(\xi^*_i, y_{j-1}), -f_y(x_{i-1}, \eta^*_j), 1\big)\big\|(x_i - x_{i-1})(y_j - y_{j-1}) \\ &= \frac{1}{2}\sqrt{1 + f_x(\xi^*_i, y_{j-1})^2 + f_y(x_{i-1}, \eta^*_j)^2}(x_i - x_{i-1})(y_j - y_{j-1}) \, . \end{aligned}$$

Similarly, there exist $\xi^{**}_i \in (x_{i-1}, x_i)$ and $\eta^{**}_j \in (y_{j-1}, y_j)$ such that the area of the triangle T^2_{ij} is given by

$$|T^2_{ij}| = \frac{1}{2}\sqrt{1 + f_x(\xi^{**}_i, y_j)^2 + f_y(x_i, \eta^{**}_j)^2}(x_i - x_{i-1})(y_j - y_{j-1}) \, .$$

Let $M = \max\limits_{(x,y)\in R}\big(\big|f_x(x,y)\big| + \big|f_y(x,y)\big|\big)$, $|R| = (b-a)(d-c)$, and $\varepsilon > 0$ be a given (but arbitrary) number. Suppose that

$$\big|f_x(\alpha,\beta) - f_x(\xi,\eta)\big| + \big|f_y(\alpha,\beta) - f_y(\xi,\eta)\big| < \frac{\varepsilon}{2|R|(1+M)} \qquad \forall\,(\alpha,\beta), (\xi,\eta) \in R_{ij} \, . \tag{14.3.1}$$

Then

$$
\begin{aligned}
&\left|\sqrt{1+f_x(\alpha,\beta)^2+f_y(\alpha^*,\beta^*)^2}-\sqrt{1+f_x(\xi,\eta)^2+f_y(\xi,\eta)^2}\right|\\
&\quad=\left|\frac{f_x(\alpha,\beta)^2+f_y(\alpha^*,\beta^*)^2-f_x(\xi,\eta)^2-f_y(\xi,\eta)^2}{\sqrt{1+f_x(\alpha,\beta)^2+f_y(\alpha^*,\beta^*)^2}+\sqrt{1+f_x(\xi,\eta)^2+f_y(\xi,\eta)^2}}\right|\\
&\quad\leqslant\frac{1}{2}\Big[\big|f_x(\alpha,\beta)-f_x(\xi,\eta)\big|\big|f_x(\alpha,\beta)+f_x(\xi,\eta)\big|\\
&\qquad\quad+\big|f_y(\alpha^*,\beta^*)-f_y(\xi,\eta)\big|\big|f_y(\alpha^*,\beta^*)+f_y(\xi,\eta)\big|\Big]\\
&\quad\leqslant\frac{2M}{2}\Big[\big|f_x(\alpha,\beta)-f_x(\xi,\eta)\big|+\big|f_y(\alpha^*,\beta^*)-f_y(\xi,\eta)\big|\Big]\leqslant\frac{M\varepsilon}{2|R|(1+M)}<\frac{\varepsilon}{2|R|}\,.
\end{aligned}
$$

Therefore, if (14.3.1) holds for all $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant m$, then for $(\xi_{ij},\eta_{ij})\in R_{ij}$, we have

$$
\begin{aligned}
&\Big||T^1_{ij}|+|T^2_{ij}|-\sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}(x_i-x_{i-1})(y_j-y_{j-1})\Big|\\
&\quad\leqslant\Big|\frac{1}{2}\sqrt{1+f_x(\xi_i^*,y_{j-1})^2+f_y(x_{i-1},\eta_j^*)^2}+\frac{1}{2}\sqrt{1+f_x(\xi_i^{**},y_j)^2+f_y(x_i,\eta_j^{**})^2}\\
&\qquad-\sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}\Big|(x_i-x_{i-1})(y_j-y_{j-1})\\
&\quad\leqslant\frac{\varepsilon}{2|R|}(x_i-x_{i-1})(y_j-y_{j-1})\,;
\end{aligned}
$$

thus if (14.3.1) holds for all $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant m$, then for $(\xi_{ij},\eta_{ij})\in R_{ij}$,

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{j=1}^{m}\big(|T^1_{ij}|+|T^2_{ij}|\big)-\sum_{i=1}^{n}\sum_{j=1}^{m}\sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}(x_i-x_{i-1})(y_j-y_{j-1})\right|\\
&\quad\leqslant\sum_{i=1}^{n}\sum_{j=1}^{m}\Big||T^1_{ij}|+|T^2_{ij}|-\sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}(x_i-x_{i-1})(y_j-y_{j-1})\Big|\\
&\quad\leqslant\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\varepsilon}{2|R|}(x_i-x_{i-1})(y_j-y_{j-1})=\frac{\varepsilon}{2}\,.
\end{aligned}
$$

Finally, we state as a fact that there exists $\delta_1>0$ such that (14.3.1) holds as long as $\|\mathcal{P}\|<\delta_1$. This property is called the ***uniform continuity*** of continuous functions on closed and bounded sets.

On the other hand, since the function $z=\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}$ is continuous on R (and R has area), it is Riemann integrable on R. Let

$$
\mathrm{I}=\iint\limits_{R}\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dA\,.
$$

Then there exists $\delta_2 > 0$ such that if $\mathcal{P}$ is a partition of R satisfying $\|\mathcal{P}\| < \delta_2$, then any Riemann sum of f for the partition $\mathcal{P}$ belongs to $\big(\mathrm{I} - \dfrac{\varepsilon}{2}, \mathrm{I} + \dfrac{\varepsilon}{2}\big)$. Therefore,

$$\Big|\sum_{i=1}^{n}\sum_{j=1}^{m}\sqrt{1 + f_x(\xi_{ij},\eta_{ij})^2 + f_y(\xi_{ij},\eta_{ij})^2}(x_i - x_{i-1})(y_j - y_{j-1}) - \mathrm{I}\Big| < \frac{\varepsilon}{2}\,.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$, and if $\mathcal{P} = \big\{R_{ij}\,\big|\,R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\big\}$ is a partition of R satisfying $\|\mathcal{P}\| < \delta$, then by choosing a collection of points $\big\{(\xi_{ij}, \eta_{ij})\big\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ such that $(\xi_{ij}, \eta_{ij}) \in R_{ij}$, we conclude that

$$\begin{aligned}
&\Big|\sum_{i=1}^{n}\sum_{j=1}^{m}\big(|T^1_{ij}| + |T^2_{ij}|\big) - \mathrm{I}\Big| \\
&\quad\leqslant \Big|\sum_{i=1}^{n}\sum_{j=1}^{m}\big(|T^1_{ij}| + |T^2_{ij}|\big) - \sum_{i=1}^{n}\sum_{j=1}^{m}\sqrt{1 + f_x(\xi_{ij},\eta_{ij})^2 + f_y(\xi_{ij},\eta_{ij})^2}(x_i - x_{i-1})(y_j - y_{j-1})\Big| \\
&\qquad + \Big|\sum_{i=1}^{n}\sum_{j=1}^{m}\sqrt{1 + f_x(\xi_{ij},\eta_{ij})^2 + f_y(\xi_{ij},\eta_{ij})^2}(x_i - x_{i-1})(y_j - y_{j-1}) - \mathrm{I}\Big| < \varepsilon\,.
\end{aligned}$$

This means that the approximation of the area of the surface S can be made arbitrarily closed to I; thus the area of the surface S must be I. In general, we have the following

Theorem 14.11

Let R be a closed region in the plane, and $f : R \to \mathbb{R}$ be a continuously differentiable function. Then the area of the surface $S = \big\{(x, y, z)\,\big|\,(x, y) \in R\,, z = f(x, y)\big\}$ is given by

$$\iint\limits_R \sqrt{1 + \|(\nabla f)(x, y)\|^2}\,dA = \iint\limits_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\,dA\,.$$

Example 14.12. Find the surface area of the sphere with radius r.

Let $f(x, y) = \sqrt{r^2 - x^2 - y^2}$ and $R = \big\{(x, y)\,\big|\,x^2 + y^2 \leqslant r^2\big\}$. Then the surface area of the sphere with radius r is given by

$$2\iint\limits_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\,dA = 2r\iint\limits_R \frac{1}{\sqrt{r^2 - x^2 - y^2}}\,dA\,.$$

Since R can also be expressed as $R = \big\{(x, y)\,\big|\, -r \leqslant x \leqslant r, -\sqrt{r^2 - x^2} \leqslant y \leqslant \sqrt{r^2 - x^2}\big\}$, the Fubini Theorem then implies that

$$\iint\limits_R \frac{1}{\sqrt{r^2 - x^2 - y^2}}\,dA = \int_{-r}^{r}\Big(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\sqrt{r^2 - x^2 - y^2}}\,dy\Big)dx\,.$$

By Theorem 5.63, we find that for each $-r < x < r$,

$$\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\sqrt{r^2-x^2-y^2}}\, dy = \arcsin \frac{y}{\sqrt{r^2-x^2}}\Big|_{y=-\sqrt{r^2-x^2}}^{y=\sqrt{r^2-x^2}} = \arcsin 1 - \arcsin(-1) = \pi\,.$$

Therefore,

$$\int_{-r}^{r} \Big(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\sqrt{r^2-x^2-y^2}}\, dy\Big) dx = \int_{-r}^{r} \pi\, dx = 2\pi r$$

which implies that the surface area of a sphere with radius r is $4\pi r^2$.

Example 14.13. In this example we consider the surface area of the upper hemi-sphere $z = \sqrt{r^2-x^2-y^2}$ that lies above the disk $R = \big\{(x,y)\,\big|\, x^2+y^2 \leqslant \sigma^2\big\}$, where $0 < \sigma < r$. Let $f(x,y) = \sqrt{r^2-x^2-y^2}$. Since R can also be expressed by

$$R = \big\{(x,y)\,\big|\, -r\sigma \leqslant x \leqslant \sigma, -\sqrt{\sigma^2-x^2} \leqslant y \leqslant \sqrt{\sigma^2-x^2}\big\}\,,$$

the Fubini Theorem implies that the surface area of interest is given by

$$\begin{aligned}
&\iint\limits_R \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\, dA \\
&\quad = \iint\limits_R \frac{r}{\sqrt{r^2-x^2-y^2}}\, dA = r\int_{-\sigma}^{\sigma} \Big(\int_{-\sqrt{\sigma^2-x^2}}^{\sqrt{\sigma^2-x^2}} \frac{1}{\sqrt{r^2-x^2-y^2}}\, dy\Big) dx\,.
\end{aligned}$$

By Theorem 5.63, we find that

$$\begin{aligned}
\int_{-\sigma}^{\sigma} \Big(\int_{-\sqrt{\sigma^2-x^2}}^{\sqrt{\sigma^2-x^2}} \frac{1}{\sqrt{r^2-x^2-y^2}}\, dy\Big) dx &= \int_{-\sigma}^{\sigma} \Big(\arcsin \frac{y}{\sqrt{r^2-x^2}}\Big|_{y=-\sqrt{\sigma^2-x^2}}^{y=\sqrt{\sigma^2-x^2}} \Big) dx \\
&= 2\int_{-\sigma}^{\sigma} \arcsin \frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-x^2}}\, dx = 2\int_{-\sigma}^{\sigma} \arctan \frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-\sigma^2}}\, dx \\
&= 2\Big[x \arctan \frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-\sigma^2}}\Big|_{x=-\sigma}^{x=\sigma} - \int_{-\sigma}^{\sigma} x \frac{d}{dx} \arctan \frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-\sigma^2}}\, dx\Big] \\
&= -2\int_{-\sigma}^{\sigma} \frac{x \cdot \frac{1}{\sqrt{r^2-\sigma^2}} \frac{-x}{\sqrt{\sigma^2-x^2}}}{1+\frac{\sigma^2-x^2}{r^2-\sigma^2}}\, dx = 2\sqrt{r^2-\sigma^2} \int_{-\sigma}^{\sigma} \frac{x^2-r^2+r^2}{(r^2-x^2)\sqrt{\sigma^2-x^2}}\, dx \\
&= -2\sqrt{r^2-\sigma^2}\pi + 2\sqrt{r^2-\sigma^2} \int_{-\sigma}^{\sigma} \frac{r^2}{(r^2-x^2)\sqrt{\sigma^2-x^2}}\, dx\,.
\end{aligned}$$

Using the substitution $x = \sigma \sin \dfrac{\theta}{2}$, we find that

$$\begin{aligned}
\int_{-\sigma}^{\sigma} \frac{r^2}{(r^2-x^2)\sqrt{\sigma^2-x^2}}\, dx &= \int_{-\pi}^{\pi} \frac{r^2}{2(r^2-\sigma^2\sin^2\frac{\theta}{2})}\, d\theta = \int_{-\pi}^{\pi} \frac{r^2}{2r^2-\sigma^2(1-\cos\theta)}\, d\theta \\
&= r^2 \int_{-\pi}^{\pi} \frac{1}{(2r^2-\sigma^2)+\sigma^2\cos\theta}\, d\theta\,.
\end{aligned}$$

and further substitution $\tan\dfrac{\theta}{2} = t$ implies that

$$\begin{aligned}
\int_{-\sigma}^{\sigma} \frac{r^2}{(r^2 - x^2)\sqrt{\sigma^2 - x^2}}\,dx &= \int_{-\infty}^{\infty} \frac{r^2}{(2r^2 - \sigma^2) + \sigma^2\frac{1-t^2}{1+t^2}}\,\frac{2dt}{1+t^2} \\
&= \int_{-\infty}^{\infty} \frac{2r^2}{2r^2(1+t^2) - \sigma^2(1+t^2) + \sigma^2(1-t^2)}\,dt \\
&= \int_{-\infty}^{\infty} \frac{r^2}{r^2 + (r^2 - \sigma^2)t^2}\,dt \\
&= \frac{r}{\sqrt{r^2 - \sigma^2}} \arctan\Big(\frac{\sqrt{r^2 - \sigma^2}}{r}t\Big)\Big|_{t=-\infty}^{\infty} = \frac{\pi r}{\sqrt{r^2 - \sigma^2}}\,.
\end{aligned}$$

Therefore, the surface area of interest is given by

$$\iint\limits_R \frac{r}{\sqrt{r^2 - x^2 - y^2}}\,dA = 2r\sqrt{r^2 - \sigma^2}\Big[-\pi + \frac{\pi r}{\sqrt{r^2 - \sigma^2}}\Big] = 2\pi r\big(r - \sqrt{r^2 - \sigma^2}\big)\,.$$

Example 14.14. Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit disk.

Let $f(x,y) = 1 + x^2 + y^2$ and $R = \big\{(x,y)\,\big|\, -1 \leqslant x \leqslant 1, -\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}\big\}$, the Fubini Theorem implies that the surface area of interest is given by

$$\iint\limits_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,dA = \int_{-1}^{1} \Big(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\Big)dx\,.$$

Using (8.3.1), we find that $\displaystyle\int \sqrt{a^2 + b^2u^2}\,du = \frac{a^2}{2b}\Big[\frac{bu\sqrt{a^2 + b^2u^2}}{a^2} + \ln\big(bu + \sqrt{a^2 + b^2u^2}\big)\Big] + C$ if $a, b > 0$; thus

$$\begin{aligned}
&\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy = 2\int_0^{\sqrt{1-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy \\
&\quad = \frac{1+4x^2}{2}\Big[\frac{2y\sqrt{1 + 4x^2 + 4y^2}}{1+4x^2} + \ln\big(2y + \sqrt{1 + 4x^2 + 4y^2}\big)\Big]\Big|_{y=0}^{y=\sqrt{1-x^2}} \\
&\quad = \sqrt{5}\sqrt{1-x^2} + \frac{1+4x^2}{2}\ln\frac{\sqrt{5} + 2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint\limits_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,dA &= \int_{-1}^{1} \Big[\sqrt{5}\sqrt{1-x^2} + \frac{1+4x^2}{2}\ln\frac{\sqrt{5} + 2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\Big]\,dx \\
&= \frac{\sqrt{5}}{2}\pi + \frac{1}{2}\int_{-1}^{1} (1+4x^2)\ln\frac{\sqrt{5} + 2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\,dx\,.
\end{aligned}$$

Integrating by parts,

$$
\begin{aligned}
&\int_{-1}^{1}(1+4x^2)\ln\frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\,dx\\
&\quad=\Big(x+\frac{4}{3}x^3\Big)\ln\frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\Big|_{x=-1}^{x=1}-\int_{-1}^{1}\Big(x+\frac{4}{3}x^3\Big)\frac{d}{dx}\ln\frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\,dx\\
&\quad=-\int_{-1}^{1}\Big(x+\frac{4}{3}x^3\Big)\frac{\sqrt{1+4x^2}}{\sqrt{5}+2\sqrt{1-x^2}}\,\frac{\frac{-2x}{\sqrt{1-x^2}}\sqrt{1+4x^2}-\frac{4x}{\sqrt{1+4x^2}}(\sqrt{5}+2\sqrt{1-x^2})}{1+4x^2}\,dx\\
&\quad=-\int_{-1}^{1}\Big(x+\frac{4}{3}x^3\Big)\frac{-2x}{\sqrt{5}+2\sqrt{1-x^2}}\,\frac{5+2\sqrt{5}\sqrt{1-x^2}}{(1+4x^2)\sqrt{1-x^2}}\,dx\\
&\quad=\frac{\sqrt{5}}{3}\int_{-1}^{1}\frac{2x(3x+4x^3)}{(1+4x^2)\sqrt{1-x^2}}\,dx=\frac{\sqrt{5}}{3}\int_{-1}^{1}\frac{-1+3(1+4x^2)-2(1-x^2)(1+4x^2)}{(1+4x^2)\sqrt{1-x^2}}\,dx\\
&\quad=\frac{-\sqrt{5}}{3}\int_{-1}^{1}\frac{1}{(1+4x^2)\sqrt{1-x^2}}\,dx+\sqrt{5}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx-\frac{2\sqrt{5}}{3}\int_{-1}^{1}\sqrt{1-x^2}\,dx\\
&\quad=\frac{-\sqrt{5}}{3}\int_{-1}^{1}\frac{1}{(1+4x^2)\sqrt{1-x^2}}\,dx+\frac{2\sqrt{5}}{3}\pi\,.
\end{aligned}
$$

By the substitution of variable $x=\sin\theta$, we find that

$$
\begin{aligned}
\int_{-1}^{1}\frac{1}{(1+4x^2)\sqrt{1-x^2}}\,dx&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+4\sin^2\theta}\,d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+2(1-\cos 2\theta)}\,d\theta\\
&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{3-2\cos 2\theta}\,d\theta=\frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{3-2\cos\phi}\,d\phi\,.
\end{aligned}
$$

By the substitution of variable $\tan\dfrac{\phi}{2}=t$, we further obtain that

$$
\begin{aligned}
\int_{-1}^{1}\frac{1}{(1+4x^2)\sqrt{1-x^2}}\,dx&=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{3-2\frac{1-t^2}{1+t^2}}\,\frac{2dt}{1+t^2}=\int_{-\infty}^{\infty}\frac{1}{1+5t^2}\,dt\\
&=\frac{1}{\sqrt{5}}\arctan(\sqrt{5}t)\Big|_{t=-\infty}^{t=\infty}=\frac{\pi}{\sqrt{5}}\,.
\end{aligned}
$$

Therefore,

$$
\iint\limits_{R}\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dA=\frac{\sqrt{5}}{2}\pi+\frac{1}{2}\Big[-\frac{\sqrt{5}}{3}\cdot\frac{\pi}{\sqrt{5}}+\frac{2\sqrt{5}\pi}{3}\Big]=\frac{\pi}{6}(5\sqrt{5}-1)\,.
$$

14.3.2 Surface area of parametric surfaces

Definition 14.15: Parametric Surfaces

Let X, Y and Z be functions of u and v that are continuous on a domain D in the uv-plane. The collection of points

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \ \text{ for some } (u,v) \in D \right\}$$

is called a parametric surface. The equations $x = X(u,v)$, $y = Y(u,v)$, and $z = Z(u,v)$ are the parametric equations for the surface, and $\boldsymbol{r} : D \to \mathbb{R}^3$ given by $\boldsymbol{r}(u,v) = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k}$ is called a parametrization of Σ.

Example 14.16. Let R be an open region in the plane, and $f : R \to \mathbb{R}$ be a continuous function. Then the graph of f is a parametric surface. In fact,

$$\text{the graph of } f = \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = \big(x, y, f(x,y)\big) \ \text{ for some } (x,y) \in R \right\}.$$

Therefore, a parametric surface can be viewed as a generalization of surfaces being graphs of functions.

Example 14.17. Let $\mathbb{S}^2 = \big\{(x,y,z) \in \mathbb{R}^3 \,\big|\, x^2 + y^2 + z^2 = 1\big\}$ be the unit sphere in $\mathbb{R}^3$. Consider

$$\boldsymbol{r}(\theta,\phi) = \big(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\big), \quad (\theta,\phi) \in D = [0,2\pi] \times [0,\pi].$$

Then $\boldsymbol{r} : D \to \mathbb{S}^2$ is a continuous bijection; thus $\mathbb{S}^2$ is a parametric surface.

Example 14.18. Consider the torus shown below

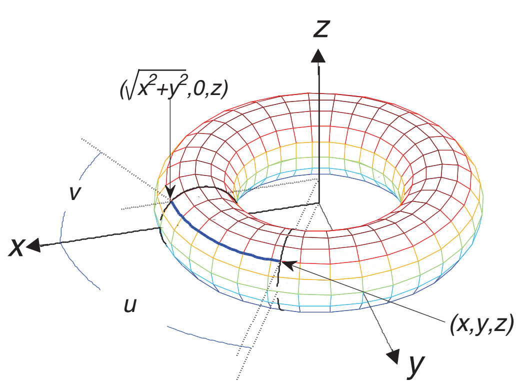


Figure 14.3: Torus with parametrization $\boldsymbol{r}(u,v)$. (temporary picture)

Note that the torus has a parametrization

$$\boldsymbol{r}(u,v) = \big((a+b\cos v)\cos u, (a+b\cos v)\sin u, b\sin v\big), \quad (u,v) \in [0,2\pi]\times[0,2\pi]\,.$$

Therefore, the torus is a parametric surface.

Remark 14.19. Similar to the case of curves, it is not required that the parametrization $\boldsymbol{r}$ is one-to-one; thus self-intersection of surface is allowed for defining parametric surface. However, we always assume that the "area" of the part of intersection is zero. This requirement is similar to the case that the parametrization of a curve that we discussed in Chapter 12 has non-overlapping property (see page 281).

Definition 14.20

A parametric surface

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}.$$

is said to be regular if X, Y, Z are differentiable funcitons and

$$\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v) \neq \mathbf{0} \qquad \forall\,(u,v) \in D\,,$$

where $\boldsymbol{r}_u \equiv X_u\mathbf{i} + Y_u\mathbf{j} + Z_u\mathbf{k}$ and $\boldsymbol{r}_v \equiv X_v\mathbf{i} + Y_v\mathbf{j} + Z_v\mathbf{k}$.

Remark 14.21. Let $\mathcal{V}$ be an open region in the plane. A vector-valued function $\psi : \mathcal{V} \to \mathbb{R}^3$ is differentiable if each component of ψ is differentiable, and the derivative of ψ, denoted by $D\psi$, is defined by

$$\big[D\psi(u,v)\big] = \begin{bmatrix} \dfrac{\partial \psi_1}{\partial u}(u,v) & \dfrac{\partial \psi_1}{\partial v}(u,v) \\ \dfrac{\partial \psi_2}{\partial u}(u,v) & \dfrac{\partial \psi_2}{\partial v}(u,v) \\ \dfrac{\partial \psi_3}{\partial u}(u,v) & \dfrac{\partial \psi_3}{\partial v}(u,v) \end{bmatrix}.$$

Therefore, a parametric surface

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}.$$

is regular if for each $(u,v) \in D$ the derivative $\big[D\psi(u,v)\big]$ has full rank.

Question: What does it mean by that a parametric surface is regular?

Suppose that

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \;\text{ for some } (u,v) \in D \right\}.$$

is regular. Then at each point $p = \boldsymbol{r}(u_0, v_0)$, $\boldsymbol{r}_u(u_0, v_0)$ and $\boldsymbol{r}_v(u_0, v_0)$ are tangent vectors to Σ so that $\boldsymbol{r}_u(u_0, v_0) \times \boldsymbol{r}_v(u_0, v_0)$ is normal to the tangent plane of Σ at p. In other words, a parametric surface is regular if every point $p \in \Sigma$ has a tangent plane (denoted by $T_p\Sigma$).

Example 14.22. Let $\mathbb{S}^2$ be the unit sphere given in Example 14.17. Then

$$\begin{aligned}
\boldsymbol{r}_\theta(\theta,\phi) &= \big(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0\big)\,,\\
\boldsymbol{r}_\phi(\theta,\phi) &= \big(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\big)
\end{aligned}$$

so that

$$\begin{aligned}
(\boldsymbol{r}_u \times \boldsymbol{r}_v)(\theta,\phi) &= \big(-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi\big)\\
&= -\sin\phi\big(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\big)
\end{aligned}$$

which is non-zero if $\phi \neq 0$ and π. Therefore, $\mathbb{S}^2\backslash\{$the north and the south poles$\}$ is a regular parametric surface (with the same parametrization except that the domain becomes $[0, 2\pi] \times (0, \pi)$).

Example 14.23. Let the torus be given in Example 14.18. Then

$$\begin{aligned}
\boldsymbol{r}_u(u,v) &= \big(-(a + b\cos v)\sin u, (a + b\cos v)\cos u, 0\big)\,,\\
\boldsymbol{r}_v(u,v) &= \big(-b\sin v\cos u, -b\sin v\sin u, b\cos v\big)
\end{aligned}$$

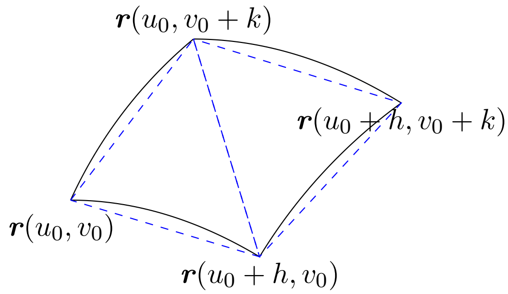
so that

$$\begin{aligned}
(\boldsymbol{r}_u \times \boldsymbol{r}_v)(u,v) &= \big(b(a+b\cos v)\cos u\cos v, b(a+b\cos v)\cos v\sin u, b(a+b\cos v)\sin v\big)\\
&= b(a+b\cos v)\big(\cos u\cos v, \sin u\cos v, \sin v\big)\,.
\end{aligned}$$

Since $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq \mathbf{0}$, we find that the torus is a regular parametric surface.

Question: How to compute the surface area of a regular parametric surface?

Let $p = \boldsymbol{r}(u_0, v_0)$ be a point in Σ, and we consider the surface area of the region $\boldsymbol{r}\big([u_0, u_0 + h] \times [v_0, v_0 + k]\big)$, where h, k are very small. This area can be approximated by the sum of the area of two triangles, one with vertices $\boldsymbol{r}(u_0, v_0)$, $\boldsymbol{r}(u_0 + h, v_0)$, $\boldsymbol{r}(u_0, v_0 + k)$ and the other with vertices $\boldsymbol{r}(u_0 + h, v_0)$, $\boldsymbol{r}(u_0, v_0 + k)$, $\boldsymbol{r}(u_0 + h, v_0 + k)$.



The area of the triangle with vertices $\boldsymbol{r}(u_0, v_0)$, $\boldsymbol{r}(u_0 + h, v_0)$, $\boldsymbol{r}(u_0, v_0 + k)$ is

$$A_1 = \frac{1}{2}\big\|\big(\boldsymbol{r}(u_0 + h, v_0) - \boldsymbol{r}(u_0, v_0)\big) \times \big(\boldsymbol{r}(u_0, v_0 + k) - \boldsymbol{r}(u_0, v_0)\big)\big\|_{\mathbb{R}^3} .$$

By the mean value theorem,

$$\begin{aligned}
&\boldsymbol{r}(u_0+h, v_0) - \boldsymbol{r}(u_0, v_0) \\
&\qquad = \big[X(u_0+h, v_0) - X(u_0, v_0)\big]\mathbf{i} + \big[Y(u_0+h, v_0) - Y(u_0, v_0)\big]\mathbf{j} \\
&\qquad\quad + \big[Z(u_0+h, v_0) - Z(u_0, v_0)\big]\mathbf{k} \\
&\qquad = h\big[X_u(u_0 + \theta_1 h, v_0)\mathbf{i} + Y_u(u_0 + \theta_2 h, v_0)\mathbf{j} + Z_u(u_0 + \theta_3 h, v_0)\mathbf{k}\big]
\end{aligned}$$

for some $\theta_1, \theta_2, \theta_3 \in (0, 1)$. Suppose that $\boldsymbol{r}$ is continuously differentiable; that is, X, Y, Z are continuously differentiable, then

$$\begin{aligned}
X_u(u_0 + \theta_1 h, v_0) &= X_u(u_0, v_0) + E_1(u_0, v_0, h)\,, \\
Y_u(u_0 + \theta_2 h, v_0) &= Y_u(u_0, v_0) + E_2(u_0, v_0, h)\,, \\
Z_u(u_0 + \theta_3 h, v_0) &= Z_u(u_0, v_0) + E_3(u_0, v_0, h)\,,
\end{aligned}$$

where E_1, E_2, E_2 approach zero as $h \to 0$. Therefore,

$$\boldsymbol{r}(u_0 + h, v_0) - \boldsymbol{r}(u_0, v_0) = h\big[\boldsymbol{r}_u(u_0, v_0) + \boldsymbol{E}_1(u_0, v_0, h)\big]\,,$$

where $\boldsymbol{E}_1 = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k}$ satisfying that $\lim\limits_{h\to 0} \boldsymbol{E}_1(u_0, v_0; h) = 0$. Similarly,

$$\boldsymbol{r}(u_0, v_0 + k) - \boldsymbol{r}(u_0, v_0) = k\big[\boldsymbol{r}_u(u_0, v_0) + \boldsymbol{E}_2(u_0, v_0, h)\big]\,,$$

where $\lim\limits_{h\to 0} \boldsymbol{E}_1(u_0, v_0; h) = 0$. The discussion above shows that

$$\lim_{(h,k)\to(0,0)} \frac{\big(\boldsymbol{r}(u_0+h, v_0) - \boldsymbol{r}(u_0, v_0)\big) \times \big(\boldsymbol{r}(u_0, v_0+k) - \boldsymbol{r}(u_0, v_0)\big)}{hk} - \boldsymbol{r}_u(u_0, v_0) \times \boldsymbol{r}_v(u_0, v_0) = \mathbf{0}$$

which further implies that

$$A_1 = \frac{1}{2}\big\|\boldsymbol{r}_u(u_0,v_0)\times\boldsymbol{r}_v(u_0,v_0)\big\|hk + \mathcal{E}_1(u_0,v_0,h,k)hk$$

for some function $\mathcal{E}_1$ which is bounded and converges to 0 as $(h,k)\to(0,0)$. Similarly, the area of the triangle with vertices $\boldsymbol{r}(u_0+h,v_0)$, $\boldsymbol{r}(u_0,v_0+k)$, $\boldsymbol{r}(u_0+h,v_0+k)$ is

$$A_2 = \frac{1}{2}\big\|\boldsymbol{r}_u(u_0,v_0)\times\boldsymbol{r}_v(u_0,v_0)\big\|hk + \mathcal{E}_2(u_0,v_0,h,k)hk$$

for some function $\mathcal{E}_2$ which is bounded and converges to 0 as $(h,k)\to(0,0)$. The two formulas for A_1 and A_2 shows that

$$\begin{aligned}&\text{the surface area of } \boldsymbol{r}\big([u_0,u_0+h]\times[v_0,v_0+k]\big)\\ &\qquad = \big\|\boldsymbol{r}_u(u_0,v_0)\times\boldsymbol{r}_v(u_0,v_0)\big\|hk + \mathcal{E}(u_0,v_0,h,k)hk\end{aligned} \tag{14.3.2}$$

for some bounded function $\mathcal{E}$ which converges to 0 as the last two variables h,k approach 0.

Now consider the surface area of $\boldsymbol{r}([a,a+L]\times[b,b+W])$. Let $\varepsilon>0$ be given. Choose $N>0$ such that

$$\big|\mathcal{E}(u,v;h,k)\big| < \frac{\varepsilon}{2LW} \quad \forall\, 0<h<\frac{L}{N}, 0<k<\frac{W}{N} \text{ and } (u,v)\in[a,a+L]\times[b,b+W]\,.$$

Denote $\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|$ by $\sqrt{\mathrm{g}}$. Then

$$\Big|\sum_{j=1}^{m}\sum_{i=1}^{n}\sqrt{\mathrm{g}\big(a+\frac{i-1}{n}L, b+\frac{j-1}{m}M\big)}\frac{L}{n}\frac{W}{m} - \int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big| < \frac{\varepsilon}{2} \quad \text{if } n,m\geqslant N\,.$$

Then for $n,m\geqslant N$, with (h,k) denoting $\big(\frac{L}{n},\frac{W}{m}\big)$ (14.3.2) implies that

$$\begin{aligned}&\Big|\text{the surface area of } \boldsymbol{r}([a,a+L]\times[b,b+W]) - \int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big|\\
&\quad = \Big|\sum_{j=1}^{m}\sum_{i=1}^{n}\text{the surface area of } \boldsymbol{r}([a+(i-1)h,a+ih]\times[b+(j-1)k,b+jk])\\
&\qquad - \int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big|\\
&\quad \leqslant \Big|\sum_{j=1}^{m}\sum_{i=1}^{n}\sqrt{\mathrm{g}\big(a+(i-1)h,b+(j-1)k\big)}hk - \int_{[a,a+L]\times[b,b+W]}\sqrt{\mathrm{g}}\,d\mathbb{A}\Big|\\
&\qquad + \Big|\sum_{j=1}^{m}\sum_{i=1}^{n}f(a+(i-1)h,b+(j-1)k;h,k)hk\Big|\\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2LW}\sum_{j=1}^{m}\sum_{i=1}^{n}hk = \varepsilon\,.\end{aligned}$$

The discussion above verifies the following

Theorem 14.24

Let D be an open region in the plane, and

$$\Sigma \equiv \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{r} = X(u,v)\mathbf{i} + Y(u,v)\mathbf{j} + Z(u,v)\mathbf{k} \text{ for some } (u,v) \in D \right\}.$$

be a regular parametric surface so that r is continuously differentiable; that is, $X_u, X_v, Y_u, Y_v, Z_u, Z_v$ are continuous. Then

$$\text{the surface area of } \Sigma = \iint\limits_D \|\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)\| \, d(u,v)\,.$$

Example 14.25. Let R be an open region in the plane, and $f : R \to \mathbb{R}$ is continuously differentiable. Then Theorem 14.24 implies that the surface area of the graph of f is given by

$$\iint\limits_R \|(\boldsymbol{r}_x \times \boldsymbol{r}_y)(x,y)\| \, d\mathbb{A}\,,$$

where the parametrization $\boldsymbol{r}$ is given by $\boldsymbol{r}(x,y) = \big(x, y, f(x,y)\big)$, $(x,y) \in R$. This formula agrees with what Theorem 14.11 provides.

Example 14.26. With the parametrization of the unit sphere $\mathbb{S}^2$ given in Example 14.22, by Theorem 14.24 the surface area of $\mathbb{S}^2$ is given by

$$\iint\limits_{[0,2\pi]\times[0,\pi]} \|(\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi)(\theta,\phi)\| d(\theta,\phi) = \int_0^\pi \Big(\int_0^{2\pi} \sin\phi \, d\theta\Big) d\phi = 4\pi\,.$$

Example 14.27. With the parametrization of the torus given in Example 14.23, by Theorem 14.24 the surface area of the torus is given by

$$\iint\limits_{[0,2\pi]\times[0,2\pi]} b(a + b\cos v)\, d(u,v) = \int_0^{2\pi} \Big(\int_0^{2\pi} (ab + b^2 \cos \boldsymbol{v})\, du\Big) dv = 4\pi^2 ab\,.$$

14.4 Triple Integrals and Applications

Let Q be a bounded region in space, and $f : Q \to \mathbb{R}$ be a non-negative function which described the point density of the region. We are interested in the mass of Q.

We start with the simple case that $Q = [a,b] \times [c,d] \times [r,s]$ is a cube. Let

$$\begin{aligned}
\mathcal{P}_x &= \{a = x_0 < x_1 < \cdots < x_n = b\}\,,\\
\mathcal{P}_y &= \{c = y_0 < y_1 < \cdots < y_m = d\}\,,\\
\mathcal{P}_z &= \{r = z_0 < z_1 < \cdots < z_p = s\}\,,
\end{aligned}$$

be partitions of $[a,b]$, $[c,d]$, $[r,s]$, respectively, and $\mathcal{P}$ be a collection of non-overlapping cubes given by

$$\mathcal{P} = \left\{R_{ijk} \,\middle|\, R_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k], 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant p\right\}.$$

Such a collection $\mathcal{P}$ is called a partition of Q, and the norm of $\mathcal{P}$ is the maximum of the length of the diagonals of all R_{ijk}; that is

$$\|\mathcal{P}\| = \max\left\{\sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2 + (z_k - z_{k-1})^2} \,\middle|\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant p\right\}.$$

A Riemann sum of f for this partition $\mathcal{P}$ is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk})(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\,.$$

The mass of Q then should be the "limit" of Riemann sums as $\|\mathcal{P}\|$ approaches zero. In general, we can remove the restrictions that f is non-negative on R and still consider the limit of the Riemann sums. We have the following

Theorem 14.28

Let $Q = [a,b] \times [c,d] \times [r,s]$ be a cube in space, and $f : Q \to \mathbb{R}$ be a function. f is said to be Riemann integrable on Q if there exists a real number I such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of Q satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to $(I - \varepsilon, I + \epsilon)$. Such a number I (is unique if it exists and) is called the ***Riemann integral*** or ***triple integral of*** f ***on*** Q and is denoted by $\displaystyle\iiint_Q f(x,y,z)\,dV$.

For general bounded region Q in space, let $r > 0$ be such that $Q \subseteq [-r,r]^3$, and we define $\displaystyle\iiint_Q f(x,y,z)\,dV$ as $\displaystyle\iiint_{[-r,r]^3} \widetilde{f}(x,y,z)\,dV$, where $\widetilde{f}$ is the zero extension of f given by

$$\widetilde{f}(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in R\,,\\ 0 & \text{if } (x,y,z) \notin R\,. \end{cases}$$

Some of the properties of double integrals in Theorem 14.4 can be restated in terms of triple integrals.

1. $\displaystyle\iiint_Q (cf)(x,y,z)\,dV = c\iiint_Q f(x,y,z)\,dV$.

2. $\displaystyle\iiint_Q (f+g)(x,y,z)\,dV = \iiint_Q f(x,y,z)\,dV + \iiint_Q g(x,y,z)\,dV$.

3. $\displaystyle\iiint_{Q_1\cup Q_2} f(x,y,z)\,dV = \iiint_{Q_1} f(x,y,z)\,dV + \iiint_{Q_2} f(x,y,z)\,dV$ for all "non-overlapping" solid regions Q_1 and Q_2.

Similar to Fubini's Theorem for the evaluation of double integrals, we have the following

Theorem 14.29: Fubini's Theorem

Let Q be a region in space, and $f : Q \to \mathbb{R}$ be continuous. If Q is given by $Q = \big\{(x,y,z)\,\big|\,(x,y)\in R, g_1(x,y)\leqslant z\leqslant g_2(x,y)\big\}$ for some region R in the xy-plane, then

$$\iiint_Q f(x,y,z)\,dV = \iint_R \Big(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\Big)\,dA\,.$$

In particular, if R is expressed by $R = \big\{(x,y)\,\big|\,a\leqslant x\leqslant b, h_1(x)\leqslant y\leqslant h_2(y)\big\}$, then

$$\iiint_Q f(x,y,z)\,dV = \int_a^b \Big[\int_{h_1(x)}^{h_2(x)} \Big(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\Big)dy\Big]\,dx\,.$$

Example 14.30. Find the volume of the region Q bounded below by the paraboloid $z = x^2+y^2$ and above by the sphere $x^2+y^2+z^2=6$.

Suppose Q is a solid region in space with uniform density 1 (or say, this region is occupied by water). Then the volume of Q is identical to the mass (in terms of its numerical value); thus we find that the volume of Q is given by $\displaystyle\iiint_Q 1\,dV$. To apply the Fubini Theorem, we need to express Q as $\big\{(x,y,z)\,\big|\,(x,y)\in R, g_1(x,y)\leqslant z\leqslant g_2(x,y)\big\}$. Nevertheless, if R is the bounded region in the plane enclosed by the curve $(x^2+y^2)^2+x^2+y^2=6$ (which in fact gives $x^2+y^2=2$), then

$$Q = \big\{(x,y,z)\,\big|\,(x,y)\in R, x^2+y^2\leqslant z\leqslant \sqrt{6-x^2-y^2}\big\}$$

and the Fubini Theorem implies that

$$\text{the volume of } Q = \int_R \Big(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} 1\, dz \Big)\, dA\,.$$

Solving for R, we find that $R = \big\{(x,y) \,\big|\, -\sqrt{2} \leqslant x \leqslant \sqrt{2}, -\sqrt{2-x^2} \leqslant y \leqslant \sqrt{2-x^2}\big\}$; thus by the Fubini Theorem we find that

$$\text{the volume of } Q = \int_{-\sqrt{2}}^{\sqrt{2}} \Big[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \Big(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} 1\, dz \Big) dy \Big] dx\,.$$

Example 14.31. Evaluate $\displaystyle\int_0^{\sqrt{\pi/2}} \Big[\int_x^{\sqrt{\pi/2}} \Big(\int_1^3 \sin(y^2)\, dz \Big) dy \Big] dx$.

Let $R = \big\{(x,y) \,\big|\, 0 \leqslant x \leqslant \sqrt{\pi/2}, x \leqslant y \leqslant \sqrt{\pi/2}\big\}$, then the domain of integration is given by

$$Q = \big\{(x,y,z) \,\big|\, 0 \leqslant x \leqslant \sqrt{\pi/2}, x \leqslant y \leqslant \sqrt{\pi/2}, 1 \leqslant z \leqslant 3\big\}$$

and the iterated integral given above is the triple integral $\displaystyle\iiint\limits_Q \sin(y^2)\, dV$.

Since R can also be expressed as $R = \big\{(x,y) \,\big|\, 0 \leqslant y \leqslant \sqrt{\pi/2}, 0 \leqslant x \leqslant y\big\}$, by the Fubini Theorem we find that

$$\begin{aligned}
&\int_0^{\sqrt{\pi/2}} \Big[\int_x^{\sqrt{\pi/2}} \Big(\int_1^3 \sin(y^2)\, dz \Big) dy \Big] dx = \iiint\limits_Q \sin(y^2)\, dV \\
&\quad = \int_0^{\sqrt{\pi/2}} \Big[\int_0^y \Big(\int_1^3 \sin(y^2)\, dz \Big) dx \Big] dy = \int_0^{\sqrt{\pi/2}} 2y \sin(y^2)\, dy = -\cos(y^2)\Big|_{y=0}^{y=\sqrt{\pi/2}} = 1\,.
\end{aligned}$$

Example 14.32. Compute the iterated integrals

$$\int_0^6 \Big[\int_{\frac{z}{2}}^3 \Big(\int_{\frac{z}{2}}^y dx \Big) dy \Big] dz + \int_0^6 \Big[\int_3^{\frac{12-z}{2}} \Big(\int_{\frac{z}{2}}^{6-y} dx \Big) dy \Big] dz\,,$$

then write the sum above as a single iterated integral in the order $dydzdx$ and $dzdydx$.

We compute the two integrals above as follows:

$$\begin{aligned}
\int_0^6 \Big[\int_{\frac{z}{2}}^3 \Big(\int_{\frac{z}{2}}^y dx \Big) dy \Big] dz &= \int_0^6 \Big[\int_{\frac{z}{2}}^3 \Big(y - \frac{z}{2}\Big)\, dy \Big] dz = \int_0^6 \Big(\frac{y^2 - yz}{2} \Big|_{y=\frac{z}{2}}^{y=3} \Big)\, dz \\
&= \frac{1}{2} \int_0^6 \Big(9 - 3z + \frac{z^2}{4}\Big)\, dz = \frac{1}{2}\Big(9z - \frac{3z^2}{2} + \frac{z^3}{12}\Big)\Big|_{z=0}^{z=6} = 9\,,
\end{aligned}$$

and

$$\int_0^6 \Big[\int_3^{\frac{12-z}{2}} \Big(\int_{\frac{z}{2}}^{6-y} dx\Big) dy\Big] dz = \int_0^6 \Big[\int_3^{\frac{12-z}{2}} \big(6 - y - \frac{z}{2}\big)\, dy\Big] dz$$

$$= \frac{1}{2}\int_0^6 \big(12y - y^2 - yz\big)\Big|_{y=3}^{y=\frac{12-z}{2}} dz$$

$$= \frac{1}{2}\int_0^6 \Big[6(12-z) - \frac{144 - 24z + z^2}{4} - \frac{(12-z)z}{2} - 36 + 9 + 3z\Big)\, dz$$

$$= \frac{1}{2}\int_0^6 \Big(72 - 6z - 36 + 6z - \frac{z^2}{4} - 6z + \frac{z^2}{2} - 27 + 3z\Big)\, dz$$

$$= \frac{1}{2}\int_0^6 \Big(9 - 3z + \frac{z^2}{4}\Big)\, dz = \frac{1}{2}\Big(9z - \frac{3z^2}{2} + \frac{z^3}{12}\Big)\Big|_{z=0}^{z=6} = 9\,.$$

Therefore, the sum of the two integrals is 18.

Let

$$Q_1 = \Big\{(x,y,z)\,\Big|\, 0 \leqslant z \leqslant 6, \frac{z}{2} \leqslant y \leqslant 3, \frac{z}{2} \leqslant x \leqslant y\Big\}\,,$$

$$Q_2 = \Big\{(x,y,z)\,\Big|\, 0 \leqslant z \leqslant 6, 3 \leqslant y \leqslant \frac{12-z}{2}, \frac{z}{2} \leqslant x \leqslant 6 - y\Big\}\,.$$

Then the Fubini Theorem implies that

$$\int_0^6 \Big[\int_{\frac{z}{2}}^3 \Big(\int_{\frac{z}{2}}^{y} dx\Big) dy\Big] dz = \iiint\limits_{Q_1} dV\,, \qquad \int_0^6 \Big[\int_3^{\frac{12-z}{2}} \Big(\int_{\frac{z}{2}}^{6-y} dx\Big) dy\Big] dz = \iiint\limits_{Q_2} dV\,.$$

Let $Q = Q_1 \cup Q_2$. Since Q_1 and Q_2 are non-overlapping solid regions (their intersection is a subset of the plane $y = 3$). Then

$$\iiint\limits_{Q_1} dV + \iiint\limits_{Q_2} dV = \iiint\limits_{Q} dV\,.$$

1. Let R be the projection of Q onto the xz-plane. Then $R = \big\{(x,z)\,\big|\, 0 \leqslant x \leqslant 3, 0 \leqslant z \leqslant 2x\big\}$ (where $z = 2x$ is the projection of the plane $x = \dfrac{z}{2}$ onto the xz-plane), and Q can also be expressed as

$$Q = \big\{(x,y,z)\,\big|\, (x,z) \in R, x \leqslant y \leqslant 6 - x\big\}\,.$$

Therefore, the volume of Q is given by

$$\int_0^3 \Big[\int_0^{2x} \Big(\int_x^{6-x} dy\Big) dz\Big] dx = \int_0^3 \Big[\int_0^{2x} (6 - 2x)\, dz\Big] dx$$

$$= \int_0^3 2x(6-2x)\, dx = \Big(6x^2 - \frac{4x^3}{3}\Big)\Big|_{x=0}^{x=3} = 54 - 36 = 18\,.$$

2. Let S be the projection of Q onto the xy-plane. Then $S = \left\{(x, y) \,\middle|\, 0 \leqslant x \leqslant 3, x \leqslant y \leqslant 6 - x\right\}$, and Q can also be expressed as

$$Q = \left\{(x, y, z) \,\middle|\, (x, y) \in S, 0 \leqslant z \leqslant 2x\right\}.$$

Therefore, the volume of Q is given by

$$\int_0^3 \Big[\int_x^{6-x} \Big(\int_0^{2x} dz\Big) dy\Big] dx = \int_0^3 \Big[\int_x^{6-x} 2x\, dy\Big]\, dx = \int_0^3 2x(6 - 2x)\, dx = 18\,.$$

14.5 Change of Variables Formula

In this section, we consider the version of substitution of variables in multiple integrals. We have used the technique of substitution of variable to evaluate the iterated integrals in, for example, Example 14.13 and 14.14; however, these substitutions of variable always assume that other variables are independent of the new variable introduced by the substitution of variable. We would like to investigate the effect of making a change of variables such as $x = r\cos\theta$, $y = r\sin\theta$ in computing the double integrals.

14.5.1 Double integrals in polar coordinates

We start our discussion with double integrals in polar coordinates. Suppose that R is the shaded region shown in Figure 14.4 and $f : R \to \mathbb{R}$ is continuous.

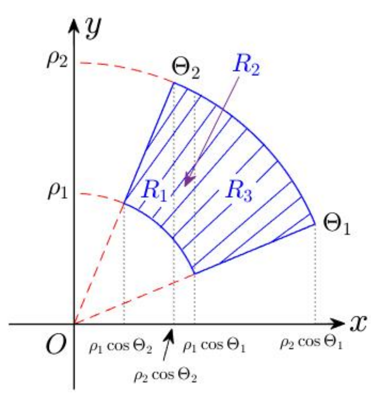


Figure 14.3: Rectangle in polar coordinates

Then to compute the double integral $\displaystyle\iint_R f(x, y)\, dA$ using the Fubini theorem directly, we need to divide R into three sub-regions R_1, R_2, R_3 given by

$$
\begin{aligned}
R_1 &= \left\{(x,y) \,\middle|\, \rho_1 \cos\Theta_2 \leqslant x \leqslant \rho_2 \cos\Theta_2, \sqrt{\rho_1^2 - x^2} \leqslant y \leqslant x \tan\Theta_2\right\},\\
R_2 &= \left\{(x,y) \,\middle|\, \rho_2 \cos\Theta_2 \leqslant x \leqslant \rho_1 \cos\Theta_1, \sqrt{\rho_2^2 - x^2} \leqslant y \leqslant \sqrt{\rho_1^2 - x^2}\right\},\\
R_3 &= \left\{(x,y) \,\middle|\, \rho_1 \cos\Theta_1 \leqslant x \leqslant \rho_2 \Theta_2, x \tan\Theta_1 \leqslant y \leqslant \sqrt{\rho_2^2 - x^2}\right\},
\end{aligned}
$$

and write

$$
\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA + \iint\limits_{R_3} f(x,y)\,dA\,.
$$

However, we know that the region R above is a rectangle in $r\theta$-plane, where (r,θ) is the polar coordinates on the plane. To be more precise, in polar coordinate the region R can be expressed as $R' \equiv \{(r,\theta) \,|\, \rho_1 \leqslant r \leqslant \rho_2, \Theta_1 \leqslant \theta \leqslant \Theta_2\}$, which means that every point (x,y) in R can be written as $(r\cos\theta, r\sin\theta)$ for $(r,\theta) \in R'$, and vice versa. One should expect that it should be easier to write down the iterated integral for computing $\iint\limits_R f(x,y)\,dA$.

Let $\mathcal{P}_r = \{\rho_1 = r_0 < r_1 < \cdots < r_n = \rho_2\}$ and $\mathcal{P}_\theta = \{\Theta_1 = \theta_0 < \theta_1 < \cdots < \theta_m = \Theta_2\}$ be partitions of $[\rho_1, \rho_2]$ and $[\Theta_1, \Theta_2]$, respectively, $R_{ij} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j]$ be rectangles in the $r\theta$-plane, S_{ij} be the sub-region in the xy-plane corresponds to R_{ij} under the polar coordinate; that is,

$$
S_{ij} = \left\{(r\cos\theta, r\sin\theta) \,\middle|\, r \in [r_{i-1}, r_i], \theta \in [\theta_{j-1}, \theta_j]\right\}.
$$

The collection $\mathcal{P} = \{S_{ij} \,|\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ is called a partition of rectangles in polar coordinates, and the norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the maximum diameter of S_{ij}.

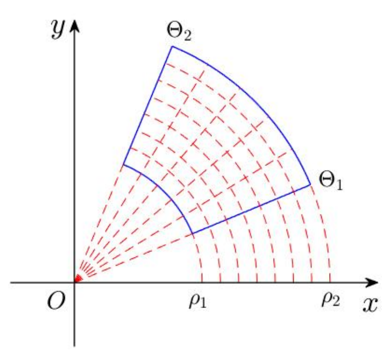


Figure 14.4: Rectangle in polar coordinates

A Riemann sum of f for partition $\mathcal{P}$ is of the form $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} f(\xi_{ij},\eta_{ij})|S_{ij}|$, where $|S_{ij}|$ is the area of S_{ij} and $\big\{(\xi_{ij},\eta_{ij})\big\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ be collection of points satisfying $(\xi_{ij},\eta_{ij}) \in S_{ij}$. Then intuitively $\iint\limits_R f(x,y)\,dA$ is the limit of Riemann sums of f for $\mathcal{P}$ as $\|\mathcal{P}\|$ approaches zero.

To see the limit of Riemann sums, we choose a particular partition $\mathcal{P}$ and collection $\big\{(\xi_{ij},\eta_{ij})\big\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$. We equally partition $[\rho_1,\rho_2]$ and $[\Theta_1,\Theta_2]$ into n and m sub-intervals. Let $\Delta r = \dfrac{\rho_2-\rho_1}{n}$ and $\Delta\theta = \dfrac{\Theta_2-\Theta_1}{m}$, and $r_i = \rho_1 + i\Delta r$ and $\theta_j = \Theta_1 + j\Delta\theta$, and $\xi_{ij} = r_i\cos\theta_j$ and $\eta_{ij} = r_i\sin\theta_j$. Noting that

$$|S_{ij}| = \frac{1}{2}(r_i^2 - r_{i-1}^2)(\theta_j-\theta_{j-1}) = \frac{1}{2}(r_i + r_{i-1})\Delta r\Delta\theta = r_i\Delta r\Delta\theta - \frac{1}{2}\Delta r^2\Delta\theta\,,$$

we find that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_{ij},\eta_{ij})|S_{ij}| &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i\cos\theta_j, r_i\sin\theta_j){\color{red}r_i}\Delta r\Delta\theta \\ &\quad - \frac{\Delta r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i\cos\theta_j, r_i\sin\theta_j)\Delta r\Delta\theta\,.\end{aligned}$$

Let $g(r,\theta) = {\color{red}r}f(r\cos\theta, r\sin\theta)$ and $h(r,\theta) = f(r\cos\theta, r\sin\theta)$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_{ij},\eta_{ij})|S_{ij}| = \sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i,\theta_j)\Delta r\Delta\theta - \frac{\Delta r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} h(r_i,\theta_j)\Delta r\Delta\theta\,.$$

As n,m approach ∞, we find that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i,\theta_j)\Delta r\Delta\theta &\to \iint\limits_{R'} g(r,\theta)\,d(r,\theta) = \iint\limits_{R'} f(r\cos\theta, r\sin\theta){\color{red}r}\,d(r,\theta)\,,\\ \sum_{i=1}^{n}\sum_{j=1}^{m} h(r_i,\theta_j)\Delta r\Delta\theta &\to \iint\limits_{R'} h(r,\theta)\,d(r,\theta) = \iint\limits_{R'} f(r\cos\theta, r\sin\theta)\,d(r,\theta)\,,\end{aligned}$$

where the right-hand side integrals denotes the double integrals on the rectangle R'. Therefore, the limit of Riemann sums of f for $\mathcal{P}$ as $\|\mathcal{P}\|$ approaches zero is

$$\iint\limits_{R'} f(r\cos\theta, r\sin\theta){\color{red}r}\,d(r,\theta)\,;$$

thus

$$\iint\limits_{R} f(x,y)\,d(x,y) = \iint\limits_{R'} f(r\cos\theta, r\sin\theta){\color{red}r}\,d(r,\theta)\,. \tag{14.5.1}$$

14.5.2 Jacobian

Recall the substitution of variables formula for the integral of functions of one variable:

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

Suppose that $g:[a,b]\to\mathbb{R}$ is one-to-one. If g is increasing, then $g'\geqslant 0$ and $g([a,b]) = \big[g(a),g(b)\big]$; thus the formula above can be rewritten as

$$\int_{g([a,b])} f(u)\,du = \int_{[a,b]} f(g(x))g'(x)\,dx = \int_{[a,b]} f(g(x))\big|g'(x)\big|\,dx\,.$$

If g is decreasing, then $g'\leqslant 0$ and $g([a,b]) = \big[g(b),g(a)\big]$; thus the formula above can be written as

$$\int_{g([a,b])} f(u)\,du = -\int_{[a,b]} f(g(x))g'(x)\,dx = \int_{[a,b]} f(g(x))\big|g'(x)\big|\,dx\,.$$

Therefore, in either cases we have a rewritten version of the substitution of variable formula

$$\int_{g([a,b])} f(u)\,du = \int_{[a,b]} f(g(x))\big|g'(x)\big|\,dx\,.$$

In this section, we are concerned with the substitution of variable formula (usually called the change of variables formula in the case of multiple integrals) for double and triple integrals, here the substitution of variables is usually given by $x = x(u,v), y = y(u,v)$ for the case of double integrals and $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ for the case of triple integrals.

Consider the double integral $\displaystyle\iint\limits_R f(x,y)\,dA$. Suppose that we have the change of variables $x = x(u,v)$ and $y = y(u,v)$, and the Fubini Theorem implies that the double integral can be written as $\displaystyle\int\Big(\int f(x,y)\,dy\Big)dx$, here we do not write the upper limit and lower limit explicitly. Note the inner integral in the iterated integral is computed by assuming that x is fixed. When x is a fixed constant, the relation $x = x(u,v)$ gives a relation between u and v, and the implicit differentiation provides that

$$\frac{du}{dv} = -\frac{x_v(u,v)}{x_u(u,v)}$$

if $x_u \neq 0$. Making the substitution of the variable $y = y(u,v)$ with u, v satisfying the relation $x = x(u,v)$, we find that

$$\begin{aligned} dy &= y_u(u,v)du + y_v(u,v)dv = y_u(u,v)\frac{du}{dv}dv + y_v(u,v)dv \\ &= \frac{x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v)}{x_u(u,v)}\, dv\,; \end{aligned}$$

thus

$$\int f(x,y)\,dy = \int f(x(u,v),y(u,v))\Big|\frac{x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v)}{x_u(u,v)}\Big|\,dv\,.$$

Therefore, the substitution of variable $x = x(u,v)$, where "v is treated as a constant since it has been integrated", is

$$\begin{aligned} &\int \Big(\int f(x,y)\,dy\Big)dx \\ &\quad= \int \Big(\int f(x(u,v),y(u,v))\Big|\frac{x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v)}{x_u(u,v)}\Big|\,dv\Big)\big|x_u(u,v)\big|\,du \\ &\quad= \int \Big(\int f(x(u,v),y(u,v))\big|x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v)\big|\,dv\Big)du\,. \end{aligned} \tag{14.5.2}$$

Example 14.33. Consider the change of variables using polar coordinate $x = r\cos\theta$, $y = r\sin\theta$ (treat r, θ as the u, v variables, respectively). Then

$$|x_u y_v - x_v y_u| = |\cos\theta\cdot r\cos\theta - (-r\sin\theta)\cdot\sin\theta| = |r| = r\,;$$

thus (14.5.2) implies the change of variables formula for polar coordinates (14.5.1).

Now we consider the possible change of variables formula for triple integrals. Suppose that by the Fubini Theorem,

$$\iiint\limits_Q f(x,y,z)\,dV = \int\Big[\int\Big(\int f(x,y,z)\,dz\Big)dy\Big]dx\,,$$

where again we do not state explicitly the upper and the lower limit of each integral. For a given change of variables $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$, the first integral that we need to evaluate is $\displaystyle\int f(x,y,z)\,dz$, and this integral is computed by assuming that x, y are fixed constants. When x and y are fixed constants, the relations $x = x(u,v,w)$ and $y = y(u,v,w)$ give a relation among u, v, w. Suppose that these relations imply that u and v

are differentiable functions of w, then the implicit differentiation (when applicable) provides that

$$\begin{aligned}0 &= x_u(u,v,w)\frac{du}{dw} + x_v(u,v,w)\frac{dv}{dw} + x_w(u,v,w)\,,\\ 0 &= y_u(u,v,w)\frac{du}{dw} + y_v(u,v,w)\frac{dv}{dw} + y_w(u,v,w)\,;\end{aligned}$$

thus if $x_u y_v - x_v y_u \neq 0$, we have

$$\begin{aligned}\frac{du}{dw} &= \frac{x_v(u,v,w)y_w(u,v,w) - x_w(u,v,w)y_v(u,v,w)}{x_u(u,v,w)y_v(u,v,w) - x_v(u,v,w)y_u(u,v,w)}\,,\\ \frac{dv}{dw} &= \frac{x_w(u,v,w)y_u(u,v,w) - x_u(u,v,w)y_w(u,v,w)}{x_u(u,v,w)y_v(u,v,w) - x_v(u,v,w)y_u(u,v,w)}\,,\end{aligned}$$

and these identities further imply that

$$\begin{aligned}dz &= z_u(u,v,w)du + z_v(u,v,w)dv + z_w(u,v,w)dw\\ &= \Big[z_u\frac{x_v y_w - x_w y_v}{x_u y_v - x_v y_u} + z_v\frac{x_w y_u - x_u y_w}{x_u y_v - x_v y_u} + z_w\Big](u,v,w)dw\\ &= \Big[\frac{x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w}{x_u y_v - x_v y_u}\Big](u,v,w)dw\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\int f(x,y,z)\,dz = \int f(x(u,v,w),y(u,v,w),z(u,v,w))\times&\\ \times\Big|\frac{x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w}{x_u y_v - x_v y_u}\Big|&(u,v,w)\,dw\,,\end{aligned}$$

and (14.5.2), by treating w as a constant since it has been integrated, implies that

$$\begin{aligned}&\int\Big[\int\Big(\int f(x,y,z)\,dz\Big)dy\Big]dx\\ &\quad= \int\Big[\int\Big(\int f(x(u,v,w),y(u,v,w),z(u,v,w))\times\\ &\qquad\qquad\times\Big|\frac{x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w}{x_u y_v - x_v y_u}\Big|(u,v,w)\,dw\Big)\times\\ &\qquad\qquad\times\big|x_u(u,v,w)y_v(u,v,w) - x_v(u,v,w)y_u(u,v,w)\big|\,dv\Big]du\\ &\quad= \int\Big[\int\Big(\int f(x(u,v,w),y(u,v,w),z(u,v,w))\times\\ &\qquad\qquad\times\big|x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w\big|(u,v,w)\,dw\Big)dv\Big]du\,.\end{aligned}$$

The naive (but **wrong**) computations above motivate the following

Definition 14.34

If $x = x(u, v)$ and $y = y(u, v)$, the ***Jacobian*** of x and y with respect to u and v, denoted by $\dfrac{\partial(x, y)}{\partial(u, v)}$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u\,.$$

If $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$, the ***Jacobian*** of x, y and z with respect to u, v and w, denoted by $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$, is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = x_u y_v z_w + x_w y_u z_v + x_v y_w z_u - x_w y_v z_u - x_v y_u z_w - x_u y_w z_v\,.$$

In general, if $g_1, g_2, \cdots, g_n$ are functions of n-variables (whose variables are denoted by $u_1, u_2, \cdots, u_n$), then the Jacobian of $g_1, g_2, \cdots, g_n$ (with respect to $u_1, u_2, \cdots, u_n$), denoted by $\dfrac{\partial(g_1, \cdots, g_n)}{\partial(u_1, \cdots, u_n)}$, is

$$\frac{\partial(g_1, \cdots, g_n)}{\partial(u_1, \cdots, u_n)} = \begin{vmatrix} \dfrac{\partial g_1}{\partial u_1} & \dfrac{\partial g_1}{\partial u_2} & \cdots & \dfrac{\partial g_1}{\partial u_n} \\ \dfrac{\partial g_2}{\partial u_1} & \dfrac{\partial g_2}{\partial u_2} & \cdots & \dfrac{\partial g_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_n}{\partial u_1} & \dfrac{\partial g_n}{\partial u_2} & \cdots & \dfrac{\partial g_n}{\partial u_n} \end{vmatrix}.$$

Example 14.35. The Jacobian of the change of variables given by the polar coordinate $x = a + r\cos\theta$, $y = b + r\sin\theta$ is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\,.$$

The Jacobian of the change of variables given by the spherical coordinate $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$ is

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} &= \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\ &= -\rho^2\cos^2\theta\sin^3\phi - \rho^2\sin^2\theta\sin\phi\cos^2\phi - \rho^2\cos^2\theta\sin\phi\cos^2\phi - \rho^2\sin^2\theta\sin^3\phi \\ &= -\rho^2\cos^2\theta\sin\phi - \rho^2\sin^2\theta\sin\phi = -\rho^2\sin\phi\,. \end{aligned}$$

The Jacobian of the change of variables given by the cylindrical coordinate $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\,.$$

Even though the derivation of the change of variables is wrong; however, the conclusion is in fact correct, and we have the following

Theorem 14.36

Let $\mathrm{O} \subseteq \mathbb{R}^2$ be an open set that has area, and $g = (g_1, g_2) : \mathrm{O} \to \mathbb{R}^2$ be an one-to-one continuously differentiable function such that g^{-1} is also continuously differentiable. Assume that the Jacobian of g_1, g_2 (with respective to their variables) does not vanish in O. If $f : g(\mathrm{O}) \to \mathbb{R}$ is integrable (on $g(\mathrm{O})$), then

$$\iint\limits_{g(\mathrm{O})} f(x,y)\,dA = \iint\limits_{\mathrm{O}} f\big(g_1(u,v), g_2(u,v)\big)\Big|\frac{\partial(g_1,g_2)}{\partial(u,v)}\Big|\,dA'\,,$$

where the integral on the right-hand side is the double integral of the function $f\big(g_1(u,v), g_2(u,v)\big)\Big|\frac{\partial(g_1,g_2)}{\partial(u,v)}\Big|$ (with variables u, v) on O.

Theorem 14.37

Let $\mathrm{O} \subseteq \mathbb{R}^3$ be an open set that has volume (that is, the constant function is Riemann integrable on O), and $g = (g_1, g_2, g_3) : \mathrm{O} \to \mathbb{R}^3$ be an one-to-one continuously differentiable function such that g^{-1} is also continuously differentiable. Assume that the Jacobian of g_1, g_2, g_3 (with respective to their variables) does not vanish in O. If $f : g(\mathrm{O}) \to \mathbb{R}$ is integrable (on $g(\mathrm{O})$), then

$$\iiint\limits_{g(\mathrm{O})} f(x,y,z)\,dV = \iiint\limits_{\mathrm{O}} f\big(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)\big)\Big|\frac{\partial(g_1,g_2,g_3)}{\partial(u,v,w)}\Big|\,dV'\,,$$

where the integral on the right-hand side is the triple integral of the function $f\big(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)\big)\Big|\frac{\partial(g_1,g_2,g_3)}{\partial(u,v,w)}\Big|$ (with variables u, v, w) on O.

Remark 14.38. Suppose that O is an open set in the plane such that the boundary of O, denoted by $\partial\mathrm{O}$, has zero area. Under suitable assumptions (for example, if the set of discontinuities of f has zero area and f is bounded above or below by a constant), we have

$$\iint\limits_{\mathrm{O}} f(x,y)\,dA = \iint\limits_{\bar{\mathrm{O}}} f(x,y)\,dA\,. \tag{14.5.3}$$

Example 14.39. Let $B = \big\{(x,y)\,\big|\,x^2+y^2<R^2\big\} - [0,1)\times\{0\}$. Then the polar coordinate $x = x(r,\theta) = r\cos\theta$ and $y = y(r,\theta) = r\cos\theta$ is an one-to-one continuously differentiable function from $\mathrm{O} \equiv (0,R)\times(0,2\pi) \to \mathbb{R}^2$ and the inverse function $r = r(x,y) = \sqrt{x^2+y^2}$ and

$$\theta = \theta(x,y) = \begin{cases} \arccos\dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y>0\,,\\ \pi & \text{if } y=0\,,\\ 2\pi - \arccos\dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y<0\,, \end{cases}$$

is also continuously differentiable (which you proved in Quiz). Therefore, the change of variables formula implies that

$$\iint\limits_{B} f(x,y)dA = \iint\limits_{(0,R)\times(0,2\pi)} f(r\cos\theta, r\sin\theta)r\,dA'\,.$$

Let $D(R) = \big\{(x,y)\,\big|\,x^2+y^2 \leqslant R^2\big\}$. Then $D = B\cup\partial B$ and $[0,R]\times[0,2\pi] = (0,R)\times(0,2\pi)\cup\partial\big[(0,R)\times(0,2\pi)\big]$; thus (14.5.3) further implies that

$$\iint\limits_{D(R)} f(x,y)dA = \iint\limits_{[0,R]\times[0,2\pi]} f(r\cos\theta, r\sin\theta)r\,dA'\,.$$

In general, if a region R, in polar coordinate, can be expressed as

$$R = \big\{(r,\theta)\,\big|\,a\leqslant\theta\leqslant b, g_1(\theta)\leqslant r\leqslant g_2(\theta)\big\}\,,$$

then

$$\iint\limits_{R} f(x,y)\,dA = \int_a^b \Big(\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\Big)\,d\theta\,,$$

while if R, in polar coordinate, can be expressed as

$$R = \big\{(r,\theta)\,\big|\,c\leqslant r\leqslant d, h_1(r)\leqslant\theta\leqslant h_2(r)\big\}\,,$$

then

$$\iint\limits_{R} f(x,y)\,dA = \int_c^d \Big(\int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta)r\,d\theta\Big)\,dr\,.$$

Example 14.40. In this example we compute the double integral $\displaystyle\iint\limits_R \sqrt{1+4x^2+4y^2}\,dA$ that appears in Example 14.14, where $R = \big\{(x,y)\,\big|\,x^2+y^2 \leqslant 1\big\}$.

Using the polar coordinate, $R = \big\{(r,\theta)\,\big|\,0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\big\}$; thus

$$\iint\limits_R \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\Big(\int_0^1 \sqrt{1+4r^2}\cdot r\,dr\Big)d\theta = \int_0^{2\pi}\Big[\frac{1}{12}(1+4r^2)^{\frac{3}{2}}\Big]\Big|_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi}(5\sqrt{5}-1)\,d\theta = 2\pi(5\sqrt{5}-1)\,.$$

Example 14.41. In this example we compute the double integral $\displaystyle\iint\limits_R \frac{r}{\sqrt{r^2-x^2-y^2}}\,dA$ that appears in Example 14.13, where $R = \big\{(x,y)\,\big|\,x^2+y^2 \leqslant \sigma^2\big\}$ with $0 < \sigma < r$.

Using the polar coordinate (here we let ρ be the radial variable instead of r since r in this integral is a fixed constant), $R = \big\{(\rho,\theta)\,\big|\,0 \leqslant \rho \leqslant \sigma, 0 \leqslant \theta \leqslant 2\pi\big\}$; thus

$$\iint\limits_R \frac{r}{\sqrt{r^2-x^2-y^2}}\,dA = \int_0^{2\pi}\Big(\int_0^{\sigma} \frac{r}{\sqrt{r^2-\rho^2}}\cdot\rho\,d\rho\Big)d\theta = \int_0^{2\pi}\Big(-r\sqrt{r^2-\rho^2}\Big)\Big|_{\rho=0}^{\rho=\sigma} d\theta$$
$$= \int_0^{2\pi}\big(r^2 - r\sqrt{r^2-\sigma^2}\big)\,d\theta = 2\pi\big(r^2 - r\sqrt{r^2-\sigma^2}\big)\,.$$

Example 14.42. Let S be the subset of the upper hemisphere $z = \sqrt{1-x^2-y^2}$ enclosed by the curve C shown in the figure below

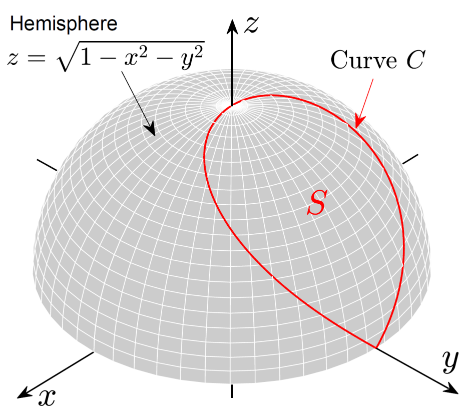


Figure 14.5: Curve S on the upper hemisphere

where each point of C corresponds to some point $(\cos t\sin t, \sin^2 t, \cos t)$ with $t \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$. Find the surface of S.

Let (x,y) be a boundary point of R. The $(x,y) = (\cos t\sin t, \sin^2 t)$ for some $t \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$; thus

$$x^2+y^2 = \cos^2 t\sin^2 t + \sin^4 t = (\cos^2 t + \sin^2 t)\sin^2 t = \sin^2 t = y\,.$$

Therefore, the boundary of R consists of points (x,y) satisfying $x^2+y^2 = y$ which shows that R is a disk centered at $\big(0,\frac{1}{2}\big)$ with radius $\frac{1}{2}$. Therefore,

$$R = \big\{(x,y)\,\big|\,0 \leqslant y \leqslant 1, -\sqrt{y-y^2} \leqslant x \leqslant \sqrt{y-y^2}\big\}\,,$$

and by Theorem 14.11 the surface area of S is given by $\displaystyle\iint\limits_R \frac{1}{\sqrt{1-x^2-y^2}}\,dA$.

Now we apply the change of variables using the polar coordinates to compute this double integral. Since we have found the Jacobian of this change of variables, we only need to find the corresponding region R' of R in the $r\theta$-plane and the change of variables formula shows that the surface area of S is $\displaystyle\iint\limits_{R'} \frac{r}{\sqrt{1-r^2}}\,dA'$.

By the fact that the boundary of R' maps to the boundary of R under the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, we find that if (r,θ) is a boundary point of R', then (r,θ) satisfies $r^2 = r\sin\theta$; thus the boundary of R' consists of points (r,θ) satisfying $r = \sin\theta$ or $r = 0$ in the $r\theta$-plane. Since R locates on the upper half plane, $0 \leqslant \theta \leqslant \pi$, and the center of the disk R corresponds to point $\big(\frac{1}{2},\frac{\pi}{2}\big)$ in the $r\theta$-plane, we conclude that

$$R' = \big\{(r,\theta)\,\big|\,0 \leqslant \theta \leqslant \pi, 0 \leqslant r \leqslant \sin\theta\big\}\,.$$

Therefore,

$$\begin{aligned}\iint\limits_{R'} \frac{r}{\sqrt{1-r^2}}\,dA' &= \int_0^\pi \Big(\int_0^{\sin\theta} \frac{1}{\sqrt{1-r^2}}\,r\,dr\Big)d\theta = \int_0^\pi \Big[\big(-\sqrt{1-r^2}\big)\Big|_{r=0}^{r=\sin\theta}\Big]\,d\theta \\ &= \int_0^\pi \big(1-|\cos\theta|\big)\,d\theta = \pi - 2\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = \pi - 2\Big(\sin\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\Big) = \pi - 2\,.\end{aligned}$$

Example 14.43. In this example we compute the improper integral $\displaystyle\int_0^\infty e^{-x^2}\,dx$. First we note that this improper integral converges since $0 \leqslant e^{-x^2} \leqslant e^{-x}$ for all $x \geqslant 1$ and $\displaystyle\int_1^\infty e^{-x}\,dx = e^{-1} < \infty$, the comparison test implies that $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges.

Let $I = \int_0^\infty e^{-x^2}\,dx$. Then $I = \int_0^\infty e^{-y^2}\,dy$; thus

$$\begin{aligned}
I^2 &= \Big(\int_0^\infty e^{-x^2}\,dx\Big)\Big(\int_0^\infty e^{-y^2}\,dy\Big) = \int_0^\infty \Big(\int_0^\infty e^{-y^2}\,dy\Big)e^{-x^2}\,dx \\
&= \int_0^\infty \Big(\int_0^\infty e^{-x^2}e^{-y^2}\,dy\Big)\,dx = \int_0^\infty \Big(\int_0^\infty e^{-(x^2+y^2)}\,dy\Big)\,dx = \iint\limits_R e^{-(x^2+y^2)}\,dA\,,
\end{aligned}$$

where R is the first quadrant of the plane. In polar coordinate, the first quadrant can be expressed as $0 < r < \infty$ and $0 < \theta < \frac{\pi}{2}$; thus using the polar coordinate we find that

$$I^2 = \int_0^{\frac{\pi}{2}} \Big(\int_0^\infty e^{-r^2} r\,dr\Big)d\theta = \int_0^{\frac{\pi}{2}} \Big(-\frac{1}{2}e^{-r^2}\Big)\Big|_{r=0}^{r=\infty}\,d\theta = \frac{\pi}{4}\,.$$

By the fact that $I \geqslant 0$, we conclude that $I = \frac{\sqrt{\pi}}{2}$.

Example 14.44. The Jacobian in the change of variable using spherical coordinate is $\rho^2\sin\phi$ Let Q be a solid region in space, and $f : Q \to \mathbb{R}$ be continuous. Suppose that Q, in spherical coordinate, can be expressed as

$$\big\{(\rho,\theta,\phi)\,\big|\, a \leqslant \phi \leqslant b, g_1(\phi) \leqslant \theta \leqslant g_2(\phi),$$

Example 14.45. In this example we reconsider the volume of Q in Example 14.30, where

$$Q = \big\{(x,y,z)\,\big|\,(x,y) \in R, x^2+y^2 \leqslant z \leqslant \sqrt{6-x^2-y^2}\big\}\,,$$

and R is a disk centered at the origin with radius $\sqrt{2}$.

Using the cylindrical coordinate, the region Q can be expressed as

$$\big\{(r,\theta,z)\,\big|\,0 \leqslant r \leqslant \sqrt{2}, 0 \leqslant \theta \leqslant 2\pi, r^2 \leqslant z \leqslant \sqrt{6-r^2}\big\}\,.$$

Therefore, the volume of Q is given by

$$\begin{aligned}
\iiint\limits_Q dV &= \int_0^{2\pi}\Big[\int_0^{\sqrt{2}}\Big(\int_{r^2}^{\sqrt{6-r^2}} r\,dz\Big)dr\Big]d\theta = \int_0^{2\pi}\Big[\int_0^{\sqrt{2}} r\big(\sqrt{6-r^2}-r^2\big)\,dr\Big]d\theta \\
&= \int_0^{2\pi}\Big[-\frac{1}{3}(6-r^2)^{\frac{3}{2}} - \frac{1}{4}r^4\Big]\Big|_{r=0}^{r=\sqrt{2}}\,d\theta = \int_0^{2\pi}\big(-\frac{8}{3}-1+2\sqrt{6}\big)\,d\theta = 2\pi\big(2\sqrt{6}-\frac{11}{3}\big)\,.
\end{aligned}$$

Example 14.46. Find the volume of the solid region Q bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=9$.

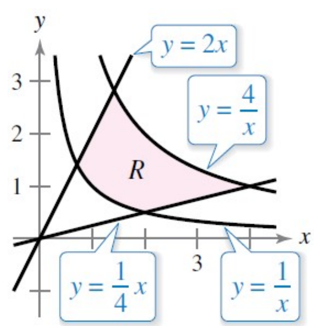
Using spherical coordinate, Q can be expressed as

$$\left\{(\rho,\theta,\phi)\,\middle|\, 0\leqslant\rho\leqslant 3, 0\leqslant\theta\leqslant 2\pi, 0\leqslant\phi\leqslant\frac{\pi}{4}\right\}.$$

Therefore, the volume of Q is given by

$$\iiint_Q dV = \int_0^{\frac{\pi}{4}}\Big[\int_0^{2\pi}\Big(\int_0^3 \rho^2\sin\phi\,d\rho\Big)d\theta\Big]d\phi = 18\pi\int_0^{\frac{\pi}{4}}\sin\phi\,d\phi = 18\pi\big(1-\frac{\sqrt{2}}{2}\big).$$

Example 14.47. Find the double integral $\displaystyle\iint_R e^{-\frac{xy}{2}}\,dA$, where R is the region given in the following figure.



Consider the following change of variables: $x=\sqrt{\frac{v}{u}}$ and $y=\sqrt{uv}$. In order to apply the change of variables formula to find the double integral, we need to know

1. What is the Jacobian of this change of variable?

2. What is the corresponding region of integration in the uv-plane?

We first note that for the change of variables to make sense, u, v have the same sign. W.L.O.G., we assume that the corresponding region in the uv-plane lies in the first quadrant. We compute the Jacobian and find that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{u}{v}}\cdot\frac{-v}{u^2} & \frac{1}{2}\sqrt{\frac{u}{v}}\cdot\frac{1}{u} \\ \frac{1}{2}\frac{v}{\sqrt{uv}} & \frac{1}{2}\frac{u}{\sqrt{uv}} \end{vmatrix} = \frac{1}{4}\cdot\frac{-1}{u}-\frac{1}{4}\cdot\frac{1}{u} = -\frac{1}{2u}.$$

Now we find the corresponding region R' in the uv-plane. The rule of thumb is that a one-to-one continuously differentiable function whose Jacobian does not vanish maps the boundary of a region to the boundary of its image. Therefore, the boundary of R' is given by $u = \frac{1}{2}$, $u = 2$ and $v = 1$, $v = 4$. Since the point (x, y) satisfying $xy = 2$ and $\frac{y}{x} = 1$ corresponds to $u = 1$ and $v = 2$, we find that $R' = \left[\frac{1}{2}, 2\right] \times [1, 4]$. Therefore, the change of variable formula implies that

$$
\begin{aligned}
\iint\limits_{R} e^{-\frac{xy}{2}}\, dA &= \iint\limits_{[\frac{1}{2},2]\times[1,4]} e^{-\frac{v}{2}} \frac{1}{2u}\, dA' = \int_{\frac{1}{2}}^{2} \Big(\int_{1}^{4} \frac{e^{-\frac{v}{2}}}{2u}\, dv \Big) du \\
&= \int_{\frac{1}{2}}^{2} \Big[\Big(\frac{-e^{-\frac{v}{2}}}{u} \Big) \Big|_{v=1}^{v=4} \Big] du = \big(e^{-\frac{1}{2}} - e^{-2}\big) \int_{\frac{1}{2}}^{2} \frac{1}{u}\, du = 3 \ln 2 \big(e^{-\frac{1}{2}} - e^{-2}\big)\,.
\end{aligned}
$$

A more fundamental question is: why do we choose this change of coordinate? The general philosophy is to "straighten" the boundary so that in the new coordinate system the corresponding region becomes a region bounded by straight lines. Observing that the boundaries of the region R consists of four curves $\frac{y}{x} = \frac{1}{4}$, $\frac{y}{x} = 2$, $xy = 1$ and $xy = 4$, it is quite intuitive that we choose $u = \frac{y}{x}$ and $v = xy$ as our change of variables (in a reverse order). Solving for x, y in terms of u, v, we find that $x = \sqrt{\frac{v}{u}}$ and $y = \sqrt{uv}$.