Calculus 微積分

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Chapter 10

Vectors and the Geometry of Space

10.1 Preliminaries

In this section we review some of the materials from the high school (or even linear algebra). First we consider vectors in the plane. We let $\mathbf{i}$ (or $\mathbf{e}_1$) and $\mathbf{j}$ (or $\mathbf{e}_2$) denote the vectors $(1,0)$ and $(0,1)$, respectively. Any vectors $\mathbf{v}$ in the plane can be written as $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$. For two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, the dot product of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}\cdot\mathbf{v}$, is defined by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 = \sum_{j=1}^{2} u_jv_j\,.$$

Let θ denote the angle between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$. The law of cosines then implies that

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\,,$$

where $\|\mathbf{u}\| = \sqrt{u_1^2+u_2^2}$ and $\|\mathbf{v}\| = \sqrt{v_1^2+v_2^2}$ denote the length of vectors $\mathbf{u}$ and $\mathbf{v}$, respectively.

Similar ideas can be extended for vectors in space. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the vectors

$$\mathbf{i} = (1,0,0) \equiv \mathbf{e}_1\,,\quad \mathbf{j} = (0,1,0) \equiv \mathbf{e}_2 \quad\text{and}\quad \mathbf{k} = (0,0,1) \equiv \mathbf{e}_3\,.$$

The standard unit vector notation for a vector $\mathbf{v}$ in space is

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = \sum_{j=1}^{3} v_j\mathbf{e}_j\,.$$

For two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the dot product of $\mathbf{u}$ and $\mathbf{v}$, again denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{j=1}^{3} u_jv_j\,.$$

If θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$ when $\mathbf{u}, \mathbf{v}$ are non-zero vectors, then the law of cosines also implies that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\,, \tag{10.1.1}$$

where $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ and $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ again denote the length of vectors $\mathbf{u}$ and $\mathbf{v}$, respectively.

10.2 The Cross Product of Two Vectors in Space

Definition 10.1

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The cross product of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\,.$$

Remark 10.2. Using the notation of determinant, the cross product of $\mathbf{u}$ and $\mathbf{v}$ can be computed as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\,.$$

Remark 10.3. A sequence $(k_1, k_2, \cdots, k_n)$ of positive integers not exceeding n, with the property that no two of the k_i are equal, is called a ***permutation of degree*** n. The collection of all permutations of degree n is denoted by $\mathbb{P}(n)$. For $1 \leqslant i, j \leqslant n$ and $i \neq j$, the operator $\tau_{(i,j)}$ interchange the i-th and j-th elements of a sequence in $\mathbb{P}(n)$. For example, if $\mathrm{n} = 3$, the permutation $(3, 1, 2)$ can be obtained by interchanging pairs of $(1, 2, 3)$ twice:

$$(1, 2, 3) \xrightarrow{\tau_{(1,3)}} (3, 2, 1) \xrightarrow{\tau_{(2,3)}} (3, 1, 2);$$

thus $(3, 1, 2)$ is called an even permutation of $(1, 2, 3)$. On the other hand, $(1, 3, 2)$ is obtained by interchanging pairs of $(1, 2, 3)$ once:

$$(1, 2, 3) \xrightarrow{\tau_{(2,3)}} (1, 3, 2);$$

thus $(1,3,2)$ is an odd permutation of $(1,2,3)$.

For $n=3$, the even and odd permutations can also be viewed as the orientation of the permutation (k_1,k_2,k_3). To be more precise, if $(1,2,3)$ is arranged in a counter-clockwise orientation (see Figure 10.1), then an even permutation of degree 3 is a permutation in the counter-clockwise orientation, while an odd permutation of degree 3 is a permutation in the clockwise orientation. From figure 10.1, it is easy to see that $(3,1,2)$ is an even permutation of degree 3 and $(1,3,2)$ is an odd permutation of degree 3.

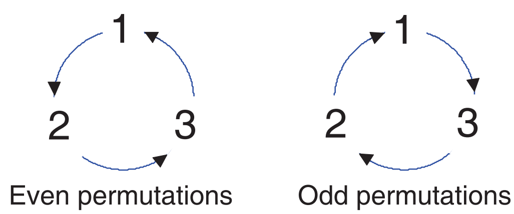

Figure 10.1: Even and odd permutations of degree 3

The permutation symbol is a function on $\mathbb{P}(n)$ defined by

$$\varepsilon_{k_1k_2\cdots k_n}=\begin{cases} 1 & \text{if } (k_1,k_2,\cdots,k_n) \text{ is an even permutation of } (1,2,\cdots,n)\,, \\ -1 & \text{if } (k_1,k_2,\cdots,k_n) \text{ is an odd permutation of } (1,2,\cdots,n)\,. \end{cases}$$

In general, one can define

$$\varepsilon_{k_1k_2\cdots k_n}=\begin{cases} 1 & \text{if } (k_1,k_2,\cdots,k_n) \text{ is an even permutation of } (1,2,\cdots,n)\,, \\ -1 & \text{if } (k_1,k_2,\cdots,k_n) \text{ is an odd permutation of } (1,2,\cdots,n)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Using the permutation symbol, we have

$$\mathbf{u}\times\mathbf{v}=\sum_{i,j,k=1}^{3}\varepsilon_{ijk}u_jv_k\mathbf{e}_i=\sum_{i=1}^{3}\Big(\sum_{j,k=1}^{3}\varepsilon_{ijk}u_jv_k\Big)\mathbf{e}_i\,, \tag{10.2.1}$$

where $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$. In other words, the i-th component of $\mathbf{u}\times\mathbf{v}$ is $\sum\limits_{j,k=1}^{3}\varepsilon_{ijk}u_jv_k$.

In the following, for simplicity we let $(\mathbf{u}\times\mathbf{v})_i$ denote the i-th component of the vector $\mathbf{u}\times\mathbf{v}$. In other words,

$$(\mathbf{u}\times\mathbf{v})_i=\sum_{j,k=1}^{3}\varepsilon_{ijk}u_jv_k\,.$$

Theorem 10.4

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in space, and c be a scalar.

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.

(c) $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$.

(d) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.

(e) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

(f) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

We note that (b) and (c) can be simplified as

$$\mathbf{u} \times (c\mathbf{v} + d\mathbf{w}) = c(\mathbf{u} \times \mathbf{v}) + d(\mathbf{u} \times \mathbf{w}) \qquad \forall \text{ vectors in space } \mathbf{u}, \mathbf{v}, \mathbf{w} \text{ and scalars } c, d.$$

Proof of Theorem 10.4. We provide two proofs for (f), and the others are left as exercise.

1. Since $\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$ and $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$, we find that
$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)\\ &= w_1(u_2v_3 - u_3v_2) + w_2(u_3v_1 - u_1v_3) + w_3(u_1v_2 - u_2v_1)\\ &= \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\,.\end{aligned}$$

2. Using (10.2.1) and the fact that $\varepsilon_{ijk} = \varepsilon_{kij}$,
$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \sum_{i=1}^{3} u_i \sum_{j,k=1}^{3} \varepsilon_{ijk} v_j w_k = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_i v_j w_k = \sum_{k=1}^{3} w_k \sum_{i,j=1}^{3} \varepsilon_{kij} u_i v_j\\ &= \sum_{k=1}^{3} w_k (\mathbf{u} \times \mathbf{v})_k = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})\,.\end{aligned}$$ □

Lemma 10.5

Let δ_{ij} be the Kronecker delta defined by $\delta_{ij} = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$ Then

$$\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}\,. \tag{10.2.2}$$

Theorem 10.6: Geometric properties of the cross product

Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

(a) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(b) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$.

(c) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

(d) $\|\mathbf{u} \times \mathbf{v}\|$ is the area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

Proof. We only prove (b). Using (10.2.2),

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 = (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = \sum_{i=1}^{3} (\mathbf{u} \times \mathbf{v})_i (\mathbf{u} \times \mathbf{v})_i &= \sum_{i=1}^{3} \Big(\sum_{j,k=1}^{3} \varepsilon_{ijk} u_j v_k \Big) \Big(\sum_{r,s=1}^{3} \varepsilon_{irs} u_r v_s \Big) \\
&= \sum_{i,j,k,r,s=1}^{3} \varepsilon_{ijk}\varepsilon_{irs} u_j v_k u_r v_s = \sum_{j,k,r,s=1}^{n} \Big(\sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{irs} \Big) u_j v_k u_r v_s \\
&= \sum_{j,k,r,s=1}^{3} (\delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}) u_j v_k u_r v_s = \sum_{j,k=1}^{3} \big[u_j^2 v_k^2 - (u_j v_j)(u_k v_k) \big] \\
&= \Big(\sum_{j=1}^{3} u_j^2 \Big) \Big(\sum_{k=1}^{3} v_k^2 \Big) - \Big(\sum_{j=1}^{3} u_j v_j \Big) \Big(\sum_{k=1}^{3} u_k v_k \Big) = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - |\mathbf{u} \cdot \mathbf{v}|^2 .
\end{aligned}$$

Using (10.1.1), we find that

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - |\mathbf{u} \cdot \mathbf{v}|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \cos^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta$$

which concludes (b). □

Definition 10.7: Triple Scalar Product

For vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in space, the dot product of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ is called the triple scalar product (of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$).

Theorem 10.8

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} .$$

Theorem 10.9

The volume V of a parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is

$$V = \left|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\right| = \left|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\right|.$$

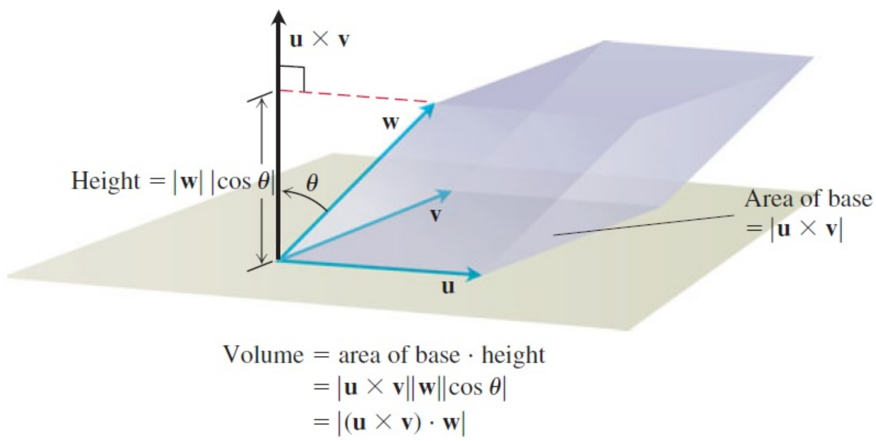

Figure 10.2: The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

10.2.1 Alternative definition of the cross product

We start with two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in space. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, they determine a plane. We select a unit vector $\boldsymbol{n}$ perpendicular to the plane by the right-hand rule; that is, the unit normal vector $\boldsymbol{n}$ points the way your right thumb points when your fingers curl through the angle θ from $\mathbf{u}$ to $\mathbf{v}$ (figure 10.3).

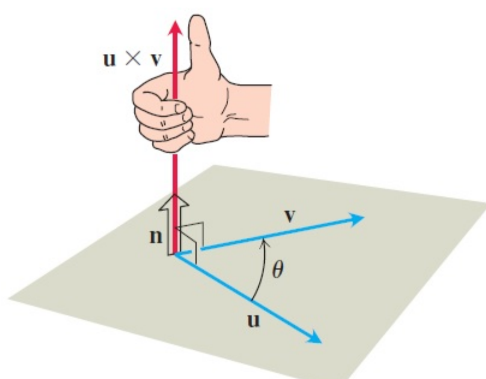

Figure 10.3: The construction of $\mathbf{u} \times \mathbf{v}$

Then we define a new vector as follows.

Definition 10.10

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in space, θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and $\boldsymbol{n}$ be a unit vector defined by the right-hand rule. The cross product $\mathbf{u} \times \mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta \, \boldsymbol{n} .$$

We note that if $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\boldsymbol{n}$ is not well-defined; however, in this case $\theta = 0$ or π so that $\sin\theta = 0$; thus the definition above suggests that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if $\mathbf{u}$ and $\mathbf{v}$ are parallel. This is indeed the case we should have in mind.

Using this definition of the cross product, properties (a)(c)(d)(e) in Theorem 10.4 clearly hold. For example, property (a) can be visualized by the following figure

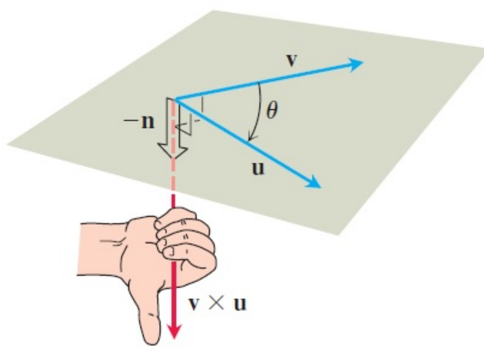

Figure 10.4: The construction of $\mathbf{u} \times \mathbf{v}$

In the following, we prove (b) in Theorem 10.4 under this alternative definition of cross product. To derive (b), we construct $\mathbf{u} \times \mathbf{v}$ in a new way (see Figure 10.5 for reference).

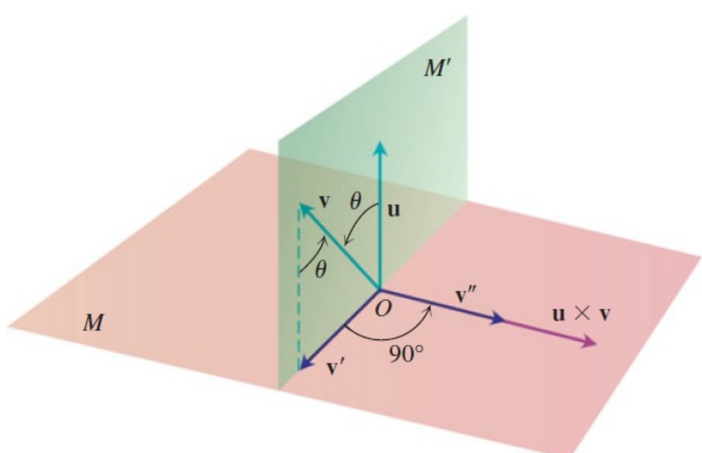

Figure 10.5: As explained in the text, $\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}''\|$. (The primes used here are purely notational and do not denote derivatives.)

We draw $\mathbf{u}$ and $\mathbf{v}$ from the common point O and construct a plane M perpendicular to $\mathbf{u}$ at O. We then project $\mathbf{v}$ orthogonally onto M, yielding a vector $\mathbf{v}'$ with length $\|\mathbf{v}\| \sin\theta$. We rotate $\mathbf{v}'$ 90° about $\mathbf{u}$ in the positive sense to produce a vector $\mathbf{v}''$. Finally, we multiply $\mathbf{v}''$ by the length of $\mathbf{u}$. The resulting vector $\|\mathbf{u}\|\mathbf{v}''$ is equal to $\mathbf{u} \times \mathbf{v}$ since $\mathbf{v}''$ has the same direction as $\mathbf{u} \times \mathbf{v}$ by its construction and

$$\|\mathbf{u}\|\|\mathbf{v}''\| = \|\mathbf{u}\|\|\mathbf{v}'\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|.$$

Now each of these three operations, namely,

1. projection onto M,
2. rotation about u through 90°,
3. multiplication by the scalar $\|\mathbf{u}\|$,

when applied to a triangle whose plane is not parallel to $\mathbf{u}$, will produce another triangle. If we start with the triangle whose sides are $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v}+\mathbf{w}$ (Figure 10.6) and apply these three steps, we successively obtain the following:

1. A triangle whose sides are $\mathbf{v}'$, $\mathbf{w}'$, and $(\mathbf{v}+\mathbf{w})'$ satisfying the vector equation
$$\mathbf{v}' + \mathbf{w}' = (\mathbf{v}+\mathbf{w})'.$$
2. A triangle whose sides are $\mathbf{v}''$, $\mathbf{w}''$, and $(\mathbf{v}+\mathbf{w})''$ satisfying the vector equation
$$\mathbf{v}'' + \mathbf{w}'' = (\mathbf{v}+\mathbf{w})''.$$
3. A triangle whose sides are $\|\mathbf{u}\|\mathbf{v}''$, $\|\mathbf{u}\|\mathbf{w}''$, and $\|\mathbf{u}\|(\mathbf{v}+\mathbf{w})''$ satisfying the vector equation
$$\|\mathbf{u}\|\mathbf{v}'' + \|\mathbf{u}\|\mathbf{w}'' = \|\mathbf{u}\|(\mathbf{v}+\mathbf{w})''.$$

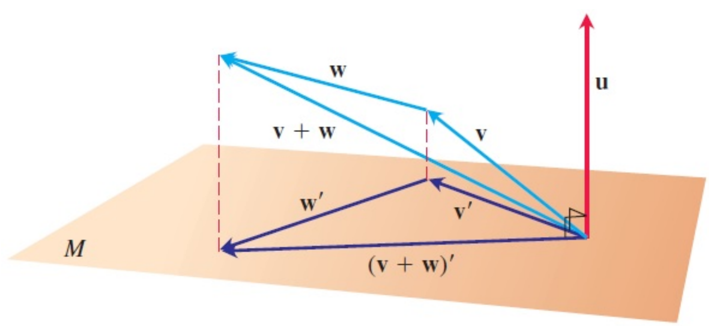

Figure 10.6: The vectors, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v}+\mathbf{w}$, and their projections onto a plane perpendicular to $\mathbf{u}$.

Substituting $\|\mathbf{u}\|\mathbf{v}'' = \mathbf{u} \times \mathbf{v}$, $\|\mathbf{u}\|\mathbf{w}'' = \mathbf{u} \times \mathbf{w}$, and $\|\mathbf{u}\|(\mathbf{v} + \mathbf{w})'' = \mathbf{u} \times (\mathbf{v} + \mathbf{w})$ from our discussion above into this last equation gives $\mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} + \mathbf{w})$, which is the law we wanted to establish.

When we apply the definition to calculate the pairwise cross products of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we find that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

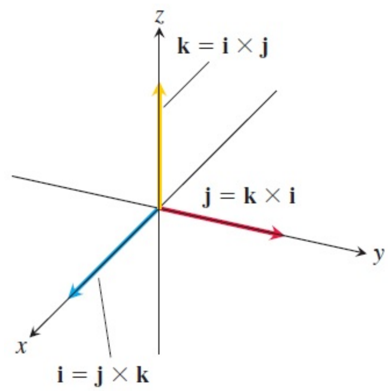

Figure 10.7: The pairwise cross products of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Having establishing (b) in Theorem 10.4 under the alternative definition of cross product, we are able to derive the formula for cross product in Definition 10.1:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= u_1v_2(\mathbf{i} \times \mathbf{j}) + u_1v_3(\mathbf{i} \times \mathbf{k}) + u_2v_1(\mathbf{j} \times \mathbf{i}) + u_2v_3(\mathbf{j} \times \mathbf{k}) + u_3v_1(\mathbf{k} \times \mathbf{i}) + u_3v_2(\mathbf{k} \times \mathbf{j}) \\ &= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\,.\end{aligned}$$

10.3 Polar Coordinate

In this section we review the polar coordinate (on the plane) that we introduction in Remark 0.8 and make some extensions. To form the polar coordinate system in the plane, fix a point O, called the pole (or origin), and construct from O an initial ray called the polar axis, as shown in Figure 10.8.

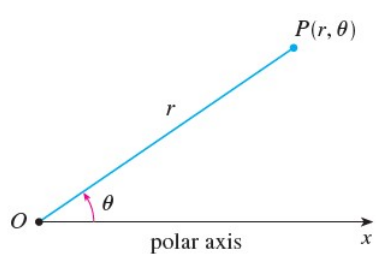

Figure 10.8: Polar coordinate

Then each point P in the plane can be assigned polar coordinates (r, θ), also called the polar representation of P, as follows.

$$\begin{aligned}
r &= \text{distance from } O \text{ to } P, \\
\theta &= \text{angle (in } [0, 2\pi)) \text{ measured counterclockwise from polar axis to segment } \overline{OP}.
\end{aligned}$$

Let the polar axis as the positive x-axis on the plane (that is, let $\mathbf{i}$ or $\mathbf{e}_1$ denote the unit vector pointing in the direction of the polar axis), and $\mathbf{j}$ or $\mathbf{e}_2$ be the unique unit vector in the plane obtained by rotating $\mathbf{i}$ counterclockwise by angle $\frac{\pi}{2}$. Then every point P in the plane can be expressed as an ordered pair (x, y) in the way that the vector $\overrightarrow{OP}$ can be expressed as $x\mathbf{e}_1 + y\mathbf{e}_2$. In other words, (x, y) is the Cartesian coordinate of P with $\mathbf{e}_1$ and $\mathbf{e}_2$ being the unit vectors on the x-axis and y-axis of the plane. If $\mathcal{P} \neq O$, and (x, y), (r, θ) are the Cartesian and polar coordinate of P, respectively, then we have

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,, \tag{10.3.1a}$$

$$r = \sqrt{x^2 + y^2}\,, \qquad \theta = \begin{cases} \arctan\dfrac{y}{x} & \text{if } x > 0\,, \\ \dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0\,, \\ \pi + \arctan\dfrac{y}{x} & \text{if } x < 0\,, \\ \dfrac{3\pi}{2} & \text{if } x = 0 \text{ and } y < 0\,. \end{cases} \tag{10.3.1b}$$

(10.3.1a) is sometimes called the **polar-to-rectangular** and (10.3.1b) is sometimes called the **rectangular-to-polar** (coordinate) conversion. Note that the polar coordinate gives an one-to-one correspondence between the region $(0, \infty) \times [0, 2\pi)$ and the plane with the origin removed.

Remark 10.11. Often time we use the region $[0, \infty) \times [0, 2\pi]$ on the $r\theta$-plane to denote the set to which (r, θ) belongs. The segment $\{0\} \times [0, 2\pi]$ is treated as the origin (of the xy-plane), while the ray $[0, \infty) \times \{0\}$ and $[0, \infty) \times \{2\pi\}$ both represent x-axis.

Such as some rectangular regions can be easily represented using the Cartesian coordinate (for example, $[a, b] \times [c, d]$ represents a rectangle), some special regions in the plane can be easily represented using the polar coordinate.

Example 10.12. The sector enclosed by the circle with radius r_0 and two radii $\theta = \theta_0$ and $\theta = \theta_1$ can be expressed as $(r, \theta) \in [0, r_0] \times [\theta_0, \theta_1]$.

Curves in the region $[0, \infty) \times [0, 2\pi]$ of the $r\theta$-plane corresponds to curves in xy-plane through relation (10.3.1a). For example, the line segment $\{1\} \times [0, 2\pi]$ (or simply $r = 1$) corresponds to the unit circle centered at the origin, and the ray $[0, \infty) \times \{\theta_0\}$ (or simply $\theta = \theta_0$) corresponds to the ray to which the angle measured from the polar axis is θ_0.

Example 10.13. The curve $r = \cos\theta$ in the region $[0, \infty) \times [0, 2\pi]$ corresponds to the circle $x^2 + y^2 = x$ in the xy-plane.

As we did not distinguish the angle 0 and 2π, we should not distinguish any θ with all $\theta + 2k\pi$ ($k \in \mathbb{Z}$). In general, for a given point $P = (x, y)$ in Cartesian coordinate system, we should treat (r, θ) as the polar coordinate of P as long as (r, θ) satisfies (10.3.1a). This includes the possibility that r is negative since

$$(r\cos\theta, r\sin\theta) = \big((-r)\cos(\theta + \pi), (-r)\sin(\theta + \pi)\big)$$

which means if (r, θ) is a polar representation of P, then $(-r, \theta + \pi)$ is also a polar representation of P.

To be more precise, the polar coordinate (r, θ) of a point P satisfies

$r =$ "directed" distance from O to P,

$\theta =$ "directed" angle measured counterclockwise from polar axis to segment OP.

We note under this convention, each point have infinitely many polar representation.

Remark 10.14. 想像你身處原點，然後你的前方是 x 軸的正方向，而座標軸上有標記單位。現在在你前方放一面鏡子，而有另一個人出現在你的後方立於座標軸上的 -2 這個位置。你所看到的是，在你的「前方」有一個位置在 -2 的人，所以你很快速地標記他的極座標為 $(-2, 0)$。在此 -2 即為所謂的 directed distance 而 0 是 directed angle。directed distance 的正負號取決於你要不要在你觀察的那個 θ 方向加一面鏡子。

From now on, the polar coordinate, given the pole and the polar axis, refers to this non-unique polar representation of points in the plane.

Theorem 10.15

The polar coordinates (r, θ) of a point are relation to the Cartesian coordinates (x, y) of the point as follows.

Polar-to-Rectangular	**Rectangular-to-Polar**
$x = r\cos\theta$	$\tan\theta = \dfrac{y}{x}$
$y = r\sin\theta$	$r^2 = x^2 + y^2$

10.4 Cylindrical and Spherical Coordinates

10.4.1 The cylindrical coordinate

Definition 10.16

In a cylindrical coordinate system, a point P in space is presented by an ordered triple (r, θ, z) such that

1. (r, θ) is a polar representation of the projection of P in the xy-plane.
2. z is the directed distance from (r, θ) to P.

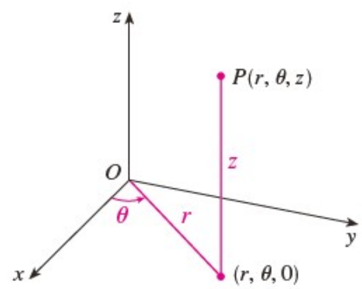

Figure 10.9: Cylindrical coordinate

The point $(0, 0, 0)$ is called the pole. Moreover, because the presentation of a point in the polar coordinate system is not unique, it follows that the representation in the cylindrical coordinate system is also not unique.

We have the coordinate conversion formula:

1. **Cylindrical to rectangular**: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
2. **Rectangular to cylindrical**: $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$, $z = z$.

10.4.2 The spherical coordinate

Definition 10.17

In a spherical coordinate system, a point P in space is represented by an ordered triple (ρ, θ, ϕ) such that

1. ρ is the distance between P and the origin (so $\rho \geqslant 0$).
2. θ is the same angle used in cylindrical coordinates for $r \geqslant 0$.
3. ϕ is the angle between the positive z-axis and the line segment $\overline{OP}$ (so $\phi \in [0, \pi]$).

Note that the first and third coordinates, ρ and ϕ, are nonnegative.

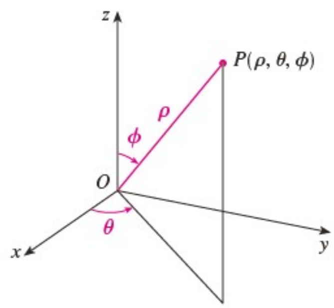

Figure 10.10: Spherical coordinate

The collection of all points whose "spherical representation" has the same $\rho > 0$ is the sphere center at the origin with radius ρ. Therefore, for fixed $\rho > 0$ the (θ, ϕ) coordinate system can be used to represent points on the sphere (centered at the origin with radius ρ) which is similar to the latitude-longitude system used to identify points on the surface of Earth. In fact, for $\rho = 6371$ kilometer (which is the radius of Earth), with the convention "north is positive and south is negative", "east is positive and west is negative", then θ is the latitude and $\frac{\pi}{2} - \phi$ is the longitude $\big($here $\theta = 0$ and $\theta = \pi$ correspond to the prime meridian (本初子午線) and the international date line (國際換日線), respectively, if $\theta \in (-\pi, \pi]\big)$.

We have the coordinate conversion formula:

1. **Spherical to rectangular**: $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$.

2. **Rectangular to spherical**: $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \dfrac{y}{x}$, $\phi = \arccos\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$.

We can also convert the spherical coordinate to cylindrical coordinate and vice versa, by the following conversion formula:

1. **Spherical to cylindrical**: $r^2 = \rho^2\sin^2\phi$, $\theta = \theta$, $z = \rho\cos\phi$.

2. **Cylindrical to spherical**: $\rho = \sqrt{r^2 + z^2}$, $\theta = \theta$, $\phi = \arccos\dfrac{z}{\sqrt{r^2 + z^2}}$.

Chapter 12

Vector-Valued Functions

12.1 Vector-Valued Functions of One Variable

Definition 12.1

A function of the form

$$\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \qquad \text{or} \qquad \boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is a vector-valued function of one variable, where the component function f, g and h are real-valued functions of the parameter t. Using the vector notation, vector-valued functions above are sometimes denoted by

$$\boldsymbol{r}(t) = \big(f(t), g(t)\big) \qquad \text{or} \qquad \boldsymbol{r}(t) = \big(f(t), g(t), h(t)\big)\,.$$

Remark 12.2. When $\boldsymbol{r}$ is a vector-valued function, we automatically assume that its components f, g (and h) have a common domain.

Definition 12.3: Limit of Vector-Valued Functions

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then
$$\lim_{t\to a} \boldsymbol{r}(t) = \Big(\lim_{t\to a} f(t)\Big)\mathbf{i} + \Big(\lim_{t\to a} g(t)\Big)\mathbf{j}$$
provided that the limits $\lim\limits_{t\to a} f(t)$ and $\lim\limits_{t\to a} g(t)$ exist.
2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then
$$\lim_{t\to a} \boldsymbol{r}(t) = \Big(\lim_{t\to a} f(t)\Big)\mathbf{i} + \Big(\lim_{t\to a} g(t)\Big)\mathbf{j} + \Big(\lim_{t\to a} h(t)\Big)\mathbf{k}$$
provided that the limits $\lim\limits_{t\to a} f(t)$, $\lim\limits_{t\to a} g(t)$ and $\lim\limits_{t\to a} h(t)$ exist.

Remark 12.4. Using the ϵ-δ language, the limit of a vector-valued function $\boldsymbol{r}$ is defined as follows: Let I be the domain of $\boldsymbol{r}$. The notation $\lim_{t\to a} \boldsymbol{r}(t) = \mathbf{L}$ means for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\boldsymbol{r}(t) - \mathbf{L}\| < \varepsilon$ whenever $0 < |t-a| < \delta$ and $t \in I$.

Definition 12.5: Continuity of Vector-Valued Functions

A vector-valued function $\boldsymbol{r}$ is said to be continuous at a point a if the limit $\lim_{t\to a} \boldsymbol{r}(t)$ exists and $\lim_{t\to a} \boldsymbol{r}(t) = \boldsymbol{r}(a)$.

Definition 12.6: Differentiation of Vector-Valued Functions

The derivative of a vector-valued function $\boldsymbol{r}$ at a point a is

$$\boldsymbol{r}'(a) = \lim_{h\to 0} \frac{\boldsymbol{r}(a+h) - \boldsymbol{r}(a)}{h}$$

provided that the limit above exists. If $\boldsymbol{r}'(a)$ exists, then $\boldsymbol{r}$ is said to be differentiable at a and $\boldsymbol{r}'(a)$ is called the derivative of $\boldsymbol{r}$ at a. If $\boldsymbol{r}'(t)$ exists for all t in an interval I, then $\boldsymbol{r}$ is said to be differentiable on the interval I.

Theorem 12.7

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then
$$\boldsymbol{r}'(a) = f'(a)\mathbf{i} + g'(a)\mathbf{j}$$
provided that $f'(a)$ and $g'(a)$ exist.
2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then
$$\boldsymbol{r}'(a) = f'(a)\mathbf{i} + g'(a)\mathbf{j} + h'(a)\mathbf{k}$$
provided that $f'(a)$, $g'(a)$ and $h'(a)$ exist.

Theorem 12.8

Let $\boldsymbol{r}$ and $\boldsymbol{u}$ be differentiable vector-valued functions and f be a differentiable real-valued function.

(a) $\dfrac{d}{dt}(f\boldsymbol{r})(t) = f'(t)\boldsymbol{r}(t) + f\boldsymbol{r}'(t)$.

(b) $\dfrac{d}{dt}\big[\boldsymbol{r}(t) \pm \boldsymbol{u}(t)\big] = \boldsymbol{r}'(t) \pm \boldsymbol{u}'(t)$.

(c) $\dfrac{d}{dt}\big[\boldsymbol{r}(t) \star \boldsymbol{u}(t)\big] = \boldsymbol{r}'(t) \star \boldsymbol{u}(t) + \boldsymbol{r}(t) \star \boldsymbol{u}'(t)$, where $\star$ is the dot product or the cross product.

(d) $\dfrac{d}{dt}\boldsymbol{r}\big(f(t)\big) = f'(t)\boldsymbol{r}'\big(f(t)\big)$.

Proof. We only prove (c) for the case $\star$ being the cross product. Write $\boldsymbol{r}(t) = r_1(t)\mathbf{i} + r_2(t)\mathbf{j} + r_3(t)\mathbf{k}$ and $\boldsymbol{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$. By the definition of the cross product, $\big[\boldsymbol{r}(t) \times \boldsymbol{u}(t)\big]_i$, the i-th component of $\boldsymbol{r}(t) \times \boldsymbol{u}(t)$, is given by $\sum\limits_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk} r_j(t) u_k(t)$. By the product rule,

$$\begin{aligned}\frac{d}{dt}\big[\boldsymbol{r}(t) \times \boldsymbol{u}(t)\big]_i &= \frac{d}{dt}\sum_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk} r_j(t) u_k(t) = \sum_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk} \frac{d}{dt}\big[r_j(t) u_k(t)\big] \\ &= \sum_{1\leqslant j,k\leqslant 3} \varepsilon_{ijk}\big[r_j'(t) u_k(t) + r_j(t) u_j'(t)\big] = \boldsymbol{r}'(t) \times \boldsymbol{u}(t) + \boldsymbol{r}(t) \times \boldsymbol{u}'(t)\,,\end{aligned}$$

where we have used $\boldsymbol{r}'(t) = r_1'(t)\mathbf{i} + r_2'(t)\mathbf{j} + r_3'(t)\mathbf{k}$ and $\boldsymbol{u}'(t) = u_1'(t)\mathbf{i} + u_2'(t)\mathbf{j} + u_3'(t)\mathbf{k}$ to conclude the last equality. $\square$

Remark 12.9. The proof presented above in fact can be used to show that

$$\begin{aligned}&\frac{d}{dt}\begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix} \\ &\quad = \begin{vmatrix} a_{11}'(t) & a_{12}'(t) & a_{13}'(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}'(t) & a_{22}'(t) & a_{23}'(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{vmatrix} + \begin{vmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}'(t) & a_{32}'(t) & a_{33}'(t)\end{vmatrix}\end{aligned}$$

since the determinant of $A = \big[a_{ij}(t)\big]_{1\leqslant i,j\leqslant 3}$ is given by $\sum\limits_{1\leqslant i,j,k\leqslant 3} \varepsilon_{ijk} a_{1i}(t) a_{2j}(t) a_{3k}(t)$. The formula above shows that the differentiation of determinants is obtained by differentiating row by row (or column by column).

• Integration of vector-valued functions of one variable

Similar to the differentiation of vector-valued functions which mimics the differentiation of real-valued functions, we can also define the Riemann integral of a vector-valued function $\boldsymbol{r}$ on $[a, b]$ as the "limit" of the Riemann sum

$$\sum_{k=1}^{n} \boldsymbol{r}(\xi_k)(t_k - t_{k-1})\,, \tag{12.1.1}$$

where $\{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$. To be more precise, a vector-valued function $\boldsymbol{r} : [a, b] \to \mathbb{R}^d$, where $d = 2$ or 3, is said to be Riemann integrable on $[a, b]$ if there exists a vector $\boldsymbol{A}$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, any Riemann sum of $\boldsymbol{r}$ for $\mathcal{P}$ (given by (12.1.1)) locates in $(\boldsymbol{A}-\varepsilon, \boldsymbol{A}+\varepsilon)$, where the vector $\boldsymbol{A}\pm\varepsilon$ is the vector obtained by adding or subtracting ε from each component of $\boldsymbol{A}$. The vector $\boldsymbol{A}$, if exists, is written as $\displaystyle\int_a^b \boldsymbol{r}(t)\,dt$. Since the limit of a vector-valued function can be computed componentwise, we have the following

Theorem 12.9

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\int_a^b \boldsymbol{r}(t)\,dt = \Big(\int_a^b f(t)dt\Big)\mathbf{i} + \Big(\int_a^b g(t)\,dt\Big)\mathbf{j}$$

provided that $\displaystyle\int_a^b f(t)\,dt$ and $\displaystyle\int_a^b g(t)\,dt$ exist.

2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\int_a^b \boldsymbol{r}(t)\,dt = \Big(\int_a^b f(t)dt\Big)\mathbf{i} + \Big(\int_a^b g(t)\,dt\Big)\mathbf{j} + \Big(\int_a^b h(t)\,dt\Big)\mathbf{k}$$

provided that $\displaystyle\int_a^b f(t)\,dt$, $\displaystyle\int_a^b g(t)\,dt$ and $\displaystyle\int_a^b h(t)\,dt$ exist.

The Fundamental Theorem of Calculus provides a way to compute the definite integral of vector-valued functions, and this enables us to define the indefinite integral of vector-valued functions as follows.

Definition 12.10

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i}+g(t)\mathbf{j}$, then the indefinite integral (anti-derivative) of $\boldsymbol{r}$ is

$$\int \boldsymbol{r}(t)\,dt = \Big(\int f(t)dt\Big)\mathbf{i} + \Big(\int g(t)\,dt\Big)\mathbf{j}$$

provided that $\displaystyle\int f(t)\,dt$ and $\displaystyle\int g(t)\,dt$ exist.

2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then the indefinite integral (anti-derivative) of $\boldsymbol{r}$ is

$$\int \boldsymbol{r}(t)\,dt = \Big(\int f(t)dt\Big)\mathbf{i} + \Big(\int g(t)\,dt\Big)\mathbf{j} + \Big(\int h(t)\,dt\Big)\mathbf{k}$$

provided that $\displaystyle\int f(t)\,dt$, $\displaystyle\int g(t)\,dt$ and $\displaystyle\int h(t)\,dt$ exist.

Having defined the indefinite integral of vector-valued functions, by the Fundamental Theorem of Calculus and Theorem 12.7 we have

$$\frac{d}{dt}\int \boldsymbol{r}(t)\,dt = \boldsymbol{r}(t)$$

as long as $\boldsymbol{r}$ is continuous.

12.2 Curves and Parametric Equations

Definition 12.11

A subset C in the plane (or space) is called a ***curve*** if C is the image of an interval $I \subseteq \mathbb{R}$ under a continuous vector-valued function $\boldsymbol{r}$. The continuous function $\boldsymbol{r} : I \to \mathbb{R}^2$ (or $\mathbb{R}^3$) is called a ***parametrization*** of the curve, and the equation

$$(x, y) = \boldsymbol{r}(t)\,,\ t \in I \qquad \big(\text{or } (x, y, z) = \boldsymbol{r}(t),\ \ t \in I\big)$$

is called a ***parametric equation*** of the curve. A curve C is called a ***plane curve*** if it is a subset in the plane.

Since a plane can be treated as a subset of space, in the following we always assume that the curve under discussion is a curve in space (so that the parametrization of the curve is given by $\boldsymbol{r} : I \to \mathbb{R}^3$).

Definition 12.12

A curve C is called ***simple*** if it has an injective parametrization; that is, there exists $\boldsymbol{r} : I \to \mathbb{R}^3$ such that $\boldsymbol{r}(I) = C$ and $\boldsymbol{r}(x) = \boldsymbol{r}(y)$ implies that $x = y$. A curve C with parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ is called ***closed*** if $I = [a, b]$ for some closed interval $[a, b] \subseteq \mathbb{R}$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. A ***simple closed*** curve C is a closed curve with parametrization $\boldsymbol{r} : [a, b] \to \mathbb{R}^3$ such that $\boldsymbol{r}$ is one-to-one on (a, b). A ***smooth*** curve C is a curve with differentiable parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ such that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

Example 12.13. The parabola $y = x^2 + 2$ on the plane is a simple smooth plane curve since $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = t\mathbf{i} + (r^2 + 2)\mathbf{j}$ is an injective differentiable parametrization of this parabola. We note that $\tilde{\boldsymbol{r}} : \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \to \mathbb{R}^2$ given by $\tilde{\boldsymbol{r}}(t) = \tan t\,\mathbf{i} + (\sec^2 t + 1)\mathbf{j}$ is also an injective smooth parametrization of this parabola. In general, a curve usually has infinitely many parameterizations.

Example 12.14. Let $I \subseteq \mathbb{R}$ be an interval, and $\boldsymbol{r} : I \to \mathbb{R}^2$ be defined by $\boldsymbol{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$. Since $\boldsymbol{r}$ is continuous and the co-domain is $\mathbb{R}^2$, the image of I under $\boldsymbol{r}$, denoted by C, is a plane curve. We note that C is part of the unit circle centered at the origin. Moreover, C is a smooth curve since $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

1. If $I = [a, b]$ and $|b - a| < 2\pi$, then C is a simple curve.

2. If $I = [0, 2\pi]$, then C is not a simple curve. However, C a simple closed curve.

Example 12.15. Let $\boldsymbol{r} : [0, 2\pi] \to \mathbb{R}^2$ be defined by $\boldsymbol{r}(t) = \sin t\,\mathbf{i} + \sin t \cos t\,\mathbf{j}$. The image $\boldsymbol{r}([0, 2\pi])$ is a curve called figure eight.

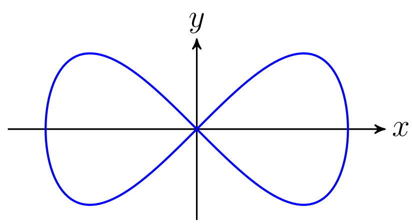

Figure 12.1: Figure eight

Example 12.16. Let $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$ be defined by $\boldsymbol{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$. Then the image $\boldsymbol{r}(\mathbb{R})$ is a simple smooth space curve. This curve is called a helix.

In the following, when a parametrization $\boldsymbol{r} : I \to \mathbb{R}^3$ of curves C is mentioned, we always assume that "there is no overlap"; that is, there are no intervals $[a, b], [c, d] \subseteq I$ satisfying that $\boldsymbol{r}([a, b]) = \boldsymbol{r}([c, d])$. If in addition

1. C is a simple curve, then $\boldsymbol{r}$ is injective, or

2. C is closed, then $I = [a, b]$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$, or

3. C is simple closed, then $I = [a, b]$ and $\boldsymbol{r}$ is injective on $[a, b)$ and $\boldsymbol{r}(a) = \boldsymbol{r}(b)$.

4. C is smooth, then $\boldsymbol{r}$ is differentiable and $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t \in I$.

12.2.1 Polar Graphs

In Example 10.13 we talk about one particular correspondence between a curve on the $r\theta$-plane and a curve on the xy-plane. The equation $r = \cos\theta$ is called a polar equation which means an equation in polar coordinate, and the corresponding curve given by the relation $(x, y) = (r\cos\theta, r\sin\theta)$ on the xy-plane is called the polar graph of this polar equation.

Definition 12.17

Let (r, θ) be the polar coordinate. A polar equation is an equation that r and θ satisfy. The polar graph of a polar equation is the collection of points $(r\cos\theta, r\sin\theta)$ in xy-plane with (r, θ) satisfying the given polar equation.

Remark 12.18. Usually, the polar equation under consideration is of the form

$$r = f(\theta) \qquad \text{or} \qquad \theta = g(r)$$

for some functions f and g. The polar graph of the polar equation $r = f(\theta)$ is the curve parameterized by the parametrization $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{r}(t) = f(t)\cos t\,\mathbf{i} + f(t)\sin t\,\mathbf{j}$ (where t is the role of θ), while the polar graph of the polar equation $\theta = g(r)$ is the curve parameterized by the parametrization $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\boldsymbol{r}(t) = t\cos g(t)\mathbf{i} + t\sin g(t)\mathbf{j}$ (where t is the role of r).

Example 12.19.
1. The polar graph of the polar equation $r = r_0$, where $r_0 \neq 0$ is a constant, is the circle centered at the origin with radius $|r_0|$.
2. The polar graph of the polar equation $\theta = \theta_0$, where θ_0 is a constant, is the straight line with slope $\tan\theta_0$.
3. The polar graph of the polar equation $r = \sec\theta$ is $x = 1$ (in the xy-plane).
4. The polar graph of the polar equation $r = a\cos\theta$, where a is a constant, is the circle centered at $\big(\frac{a}{2}, 0\big)$ with radius $\frac{|a|}{2}$.
5. The polar graph of the polar equation $r = a\sin\theta$, where a is a constant, is the circle centered at $\big(0, \frac{a}{2}\big)$ with radius $\frac{|a|}{2}$.

Example 12.20. A conic section (圓錐曲線) can be defined purely in terms of plane geometry: it is the locus of all points P whose distance to a fixed point F (called the focus 焦點) is a constant multiple (called the eccentricity e 離心率) of the distance from P to a fixed line L (called the directrix 準線). For $0 < \mathrm{e} < 1$ we obtain an ellipse, for $\mathrm{e} = 1$ a parabola, and for $\mathrm{e} > 1$ a hyperbola.

Now we consider the polar equation whose polar graph represents a conic section. Let the focus be the pole of a polar coordinate, and the polar axis is perpendicular to the directrix but does not intersect the directrix. Then the eccentricity e is given by

$$\mathrm{e} = \frac{d(P, F)}{d(P, L)} \qquad \text{for all points } P \text{ on the conic section,} \tag{12.2.1}$$

where $d(P,F)$ is the distance between P and the focus F, and $d(P,L)$ is the distance between P and the directrix.

Let P denote the distance between the pole and the directrix, and the polar coordinate of points P on a conic section is (r,θ). Then (12.2.1) implies that

$$\mathrm{e} = \frac{r}{r\cos\theta + \mathrm{P}}\,.$$

Therefore, the polar equation of a conic section with eccentricity e is given by

$$r = \frac{\mathrm{eP}}{1-\mathrm{e}\cos\theta}\,.$$

In general, for a given conic section we let the principal ray denote the ray starting from the focus, perpendicular to the directrix without intersecting the directrix. Let the focus F be the pole of a polar coordinate and θ_0 be the directed angel from the polar axis to the principal ray. If (r,θ) is the polar representation of point P on the conic section, then (r,θ) satisfies

$$\mathrm{e} = \frac{r}{r\cos(\theta-\theta_0)+\mathrm{P}} \qquad \text{or equivalently,} \quad r = \frac{\mathrm{eP}}{1-\mathrm{e}\cos(\theta-\theta_0)}\,.$$

Example 12.21 (Limaçons - 蚶線)**.** The polar graph of the polar equation $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$, where $a, b > 0$ are constants, is called a limaçon. A limaçons is also called a cardioid（心臟線）if $a = b$.

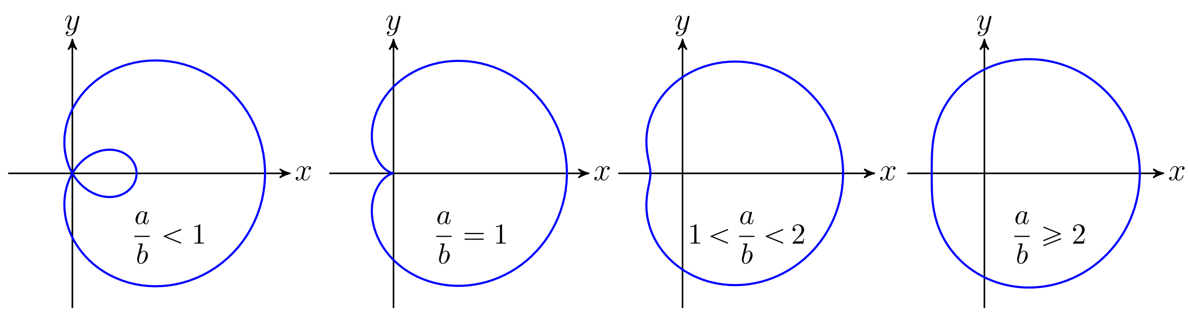

Figure 12.2: Limaçons $r = a \pm b\cos\theta$ with the ratio $\frac{a}{b}$ in different regions

(1) There is an inner loop when $\frac{a}{b} < 1$. (2) When $a = b$ it is also called the cardioid.
(3) When $1 < \frac{a}{b} < 2$, the region enclosed by the limaçon is not convex. This kind of limaçons is called dimpled limaçons. (4) When $\frac{a}{b} \geqslant 2$, it is called convex limaçons.

Example 12.22 (Rose curves)**.** The polar graph of the polar equation $r = a\cos n\theta$ or $r = a\sin n\theta$, where $a > 0$ is a given number and $n \geqslant 2$ is an integer, is called a rose curve.

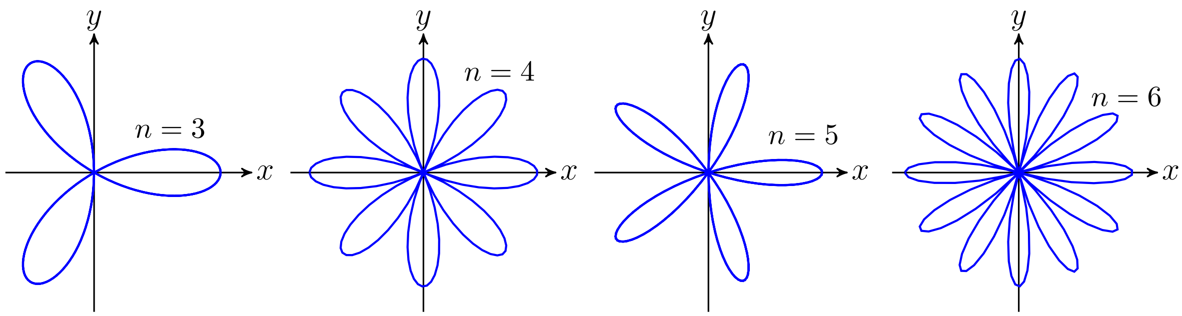

Figure 12.3: Rose curves $r = a\cos n\theta$: n petals when n is odd and $2n$ petals when n is even

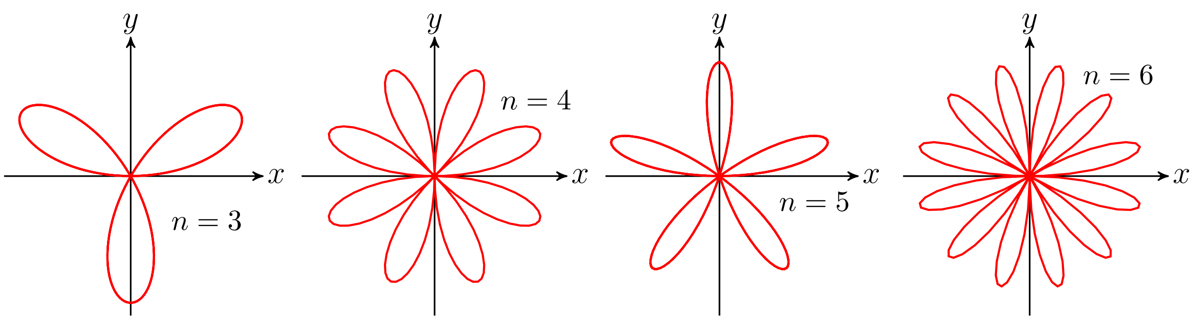

Figure 12.4: Rose curves $r = a\sin n\theta$: n petals when n is odd and $2n$ petals when n is even

Example 12.23 (Lemniscates - 雙紐線)**.** The polar graph of the polar equation $r^2 = a^2\sin 2\theta$ or $r^2 = a^2\cos 2\theta$ is called a lemniscate.

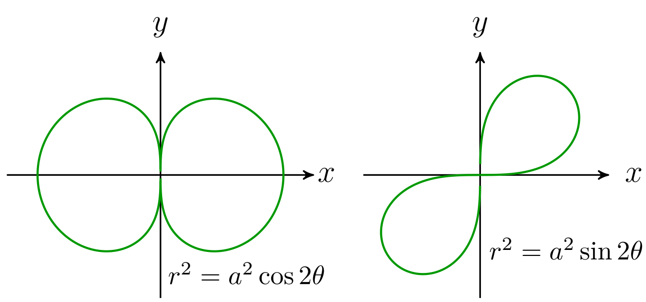

Figure 12.5: Lemniscate $r^2 = a^2\cos 2\theta$ or $r^2 = a^2\sin 2\theta$

12.3 Physical and Geometric Meanings of the Derivative of Vector-Valued Functions

Let $I \subseteq \mathbb{R}$ be an interval and $\boldsymbol{r} : I \to \mathbb{R}^3$ be a differentiable vector-valued function.

12.3.1 Physical meaning

Treat I as the time interval, and $\boldsymbol{r}(t)$ as the position of an object at time t. For $a, b \in I$ and

$a < b$, $\dfrac{\boldsymbol{r}(b) - \boldsymbol{r}(a)}{b - a}$ is the average velocity of the object in the time interval $[a, b]$. Therefore,

$$\boldsymbol{r}'(c) = \lim_{h \to 0} \frac{\boldsymbol{r}(c+h) - \boldsymbol{r}(c)}{h},$$

is the instantaneous velocity at $t = c$, and $\|\boldsymbol{r}'(c)\|$ is the instantaneous speed at $t = c$. If $\boldsymbol{r}$ is twice differentiable, then the derivative of the velocity vector $\boldsymbol{r}'$ is the acceleration.

Definition 12.24

Let $I \subseteq \mathbb{R}$ be the time interval and $\boldsymbol{r} : I \to \mathbb{R}^3$ be the position vector. The velocity vector, acceleration vector and the speed at time t are

$$\begin{aligned} \textbf{Velocity} &= \boldsymbol{v}(t) = \boldsymbol{r}'(t), \\ \textbf{Acceleration} &= \boldsymbol{a}(t) = \boldsymbol{r}''(t), \\ \textbf{Speed} &= \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\|. \end{aligned}$$

Example 12.25. Suppose a satellite is under uniform circular motion（等速率圓周運動）and the position of the satellite is given by

$$\boldsymbol{r}(t) = \big(R\cos(\omega t), R\sin(\omega t)\big),$$

where R is the distance between the satellite and the center of Earth, and ω is the angular velocity. Then

$$\boldsymbol{r}'(t) = R\omega\big(-\sin(\omega t), \cos(\omega t)\big) \quad \text{and} \quad \boldsymbol{r}''(t) = -R\omega^2\big(\cos(\omega t), \sin(\omega t)\big);$$

thus

$$\|\boldsymbol{a}(t)\| = \|\boldsymbol{r}''(t)\| = R\omega^2 = \frac{\|\boldsymbol{r}'(t)\|^2}{R} = \frac{\|\boldsymbol{v}(t)\|^2}{R}$$

which gives the famous formula for the centripetal acceleration（向心加速度）.

Example 12.26. In this example we consider the motion of a planet around a single sun. In the plane on which the planet moves, we introduce a polar coordinate system and a Cartesian coordinate system as follows:

1. Let the sun be the pole of the polar coordinate system, and fixed a polar axis on this plane.

2. Let $\mathbf{i}$ be the unit vector in the direction of the polar axis, and $\mathbf{j}$ be the corresponding unit vector obtained by rotating $\mathbf{i}$ counterclockwise by $\frac{\pi}{2}$.

Suppose the position of the planet on the plane at time $t \in I$ is given by $\boldsymbol{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. For each $t \in I$, let $(r(t), \theta(t))$ be the polar representation of $(x(t), y(t))$ in the trajectory. We would like to determine the relation that $r(t)$ and $\theta(t)$ satisfy.

Define two vectors $\widehat{r}(t) = \cos\theta(t)\mathbf{i} + \sin\theta(t)\mathbf{j}$ and $\widehat{\theta}(t) = -\sin\theta(t)\mathbf{i} + \cos\theta(t)\mathbf{j}$. Then $\boldsymbol{r} = r\widehat{r}$. Moreover, let M and m be the mass of the sun and the planet, respectively. Then Newton's second law of motion implies that

$$-\frac{GMm}{r^2}\widehat{r} = m\boldsymbol{r}''. \tag{12.3.1}$$

By the fact that $\widehat{r}' = \theta'\widehat{\theta}$ and $\widehat{\theta}' = -\theta'\widehat{r}$, we find that

$$\begin{aligned}\boldsymbol{r}'' &= \frac{d}{dt}\big(r'\widehat{r} + r\theta'\widehat{\theta}\big) = r''\widehat{r} + r'\theta'\widehat{\theta} + r'\theta'\widehat{\theta} + r\theta''\widehat{\theta} - r(\theta')^2\widehat{r} \\ &= \big[r'' - r(\theta')^2\big]\widehat{r} + \big[2r'\theta' + r\theta''\big]\widehat{\theta}.\end{aligned}$$

Therefore, (12.3.1) implies that

$$-\frac{GM}{r^2}\widehat{r} = \big[r'' - r(\theta')^2\big]\widehat{r} + \big[2r'\theta' + r\theta''\big]\widehat{\theta}.$$

Since $\widehat{r}$ and $\widehat{\theta}$ are linearly independent, we must have

$$-\frac{GM}{r^2} = r'' - r(\theta')^2, \tag{12.3.2a}$$

$$2r'\theta' + r\theta'' = 0. \tag{12.3.2b}$$

Note that (12.3.2b) implies that $(r^2\theta')' = 0$; thus $r^2\theta'$ is a constant. Since $mr^2\theta'$ is the angular momentum, (12.3.2b) implies that the angular momentum is a constant, so-called the conservation of angular momentum（角動量守恆）.

12.3.2 Geometric meaning

Suppose that the image $r(I)$ is a curve C. Since $\boldsymbol{r}(c+h) - \boldsymbol{r}(c)$ is the vector pointing from $\boldsymbol{r}(c)$ to $\boldsymbol{r}(c+h)$, we expect that $\boldsymbol{r}'(c)$, if it is not zero, is tangent to the curve at the point $\boldsymbol{r}(c)$. This motivates the following

Definition 12.27

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I. The unit tangent vector $\mathbf{T}$ (associated with the parametrization $\boldsymbol{r}$) is defined as

$$\mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}.$$

Remark 12.28. Since there are infinitely many parameterizations of a given smooth curve, different parameterizations of a smooth curve might provide different unit tangent vector. However, this is not the case and there are only two unit tangent vectors.

Theorem 12.29

Let $I \subseteq \mathbb{R}$ be an interval, and $\boldsymbol{r} : I \to \mathbb{R}^3$ be a differentiable vector-valued function. If $\|\boldsymbol{r}(t)\|$ is a constant function on I, then

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 0 \qquad \forall\, t \in I\,.$$

Proof. Suppose that $\|\boldsymbol{r}(t)\| = C$ for some constant C. Since $\|\boldsymbol{r}(t)\|^2 = \boldsymbol{r}(t) \cdot \boldsymbol{r}(t)$,

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}(t) = C^2 \qquad \forall\, t \in I\,;$$

thus by the fact that $\boldsymbol{r}$ is differentiable, Theorem 12.8 implies that

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = \frac{1}{2}\Big[\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) + \boldsymbol{r}'(t) \cdot \boldsymbol{r}(t)\Big] = \frac{1}{2}\frac{d}{dt}\big[\boldsymbol{r}(t) \cdot \boldsymbol{r}(t)\big] = 0 \qquad \forall\, t \in I\,. \qquad \square$$

Corollary 12.30

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and $\mathbf{T}(t) = \dfrac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$ be the unit tangent vector (associated with the parametrization r). If $\boldsymbol{T}$ is differentiable at t, then

$$\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0 \qquad \forall\, t \in I\,.$$

Definition 12.31

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and $\mathbf{T}(t) = \dfrac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$ be the unit tangent vector (associated with r). If $\mathbf{T}'(t)$ exists and $\mathbf{T}'(t) \neq \mathbf{0}$, then the ***principal unit normal vector*** (associated with the parametrization $\boldsymbol{r}$) at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}\,.$$

Theorem 12.32

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and the principal unit normal vector $\mathbf{N}$ exists, then the acceleration vector $\boldsymbol{a}$ lies in the plane determined by the unit tangent vector $\mathbf{T}$ and $\mathbf{N}$.

Proof. Let $\boldsymbol{v} = \boldsymbol{r}'$ be the velocity vector. Then

$$\boldsymbol{v} = \|\boldsymbol{v}\|\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \|\boldsymbol{v}\|\frac{\boldsymbol{r}'}{\|\boldsymbol{r}'\|} = \|\boldsymbol{v}\|\mathbf{T}\,.$$

Therefore,

$$\boldsymbol{a} = \boldsymbol{v}' = \|\boldsymbol{v}\|'\mathbf{T} + \|\boldsymbol{v}\|\mathbf{T}' = \|\boldsymbol{v}\|'\mathbf{T} + \|\boldsymbol{v}\|\|\mathbf{T}'\|\mathbf{N}\,.$$

The equation above implies that $\boldsymbol{a}$ is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$, it follows that $\boldsymbol{a}$ lies in the plane determined by $\mathbf{T}$ and $\mathbf{N}$. □

Remark 12.33. The coefficients of $\mathbf{T}$ and $\mathbf{N}$ in the proof above are called the ***tangential and normal components of acceleration*** and are denoted by

$$a_{\mathbf{T}} = \|\boldsymbol{v}\|' \qquad \text{and} \qquad a_{\mathbf{N}} = \|\boldsymbol{v}\|\|\mathbf{T}'\|$$

so that $\boldsymbol{a}(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t)$. Moreover, we note that the formula for $a_{\mathbf{N}}$ above shows that $a_{\mathbf{N}} \geqslant 0$. The normal component of acceleration is also called the ***centripetal component of acceleration***.

The following theorem provides some convenient formulas for computing $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$.

Theorem 12.34

Let C be a smooth curve represented by $\boldsymbol{r}$ on an interval I, and the principal unit normal vector $\mathbf{N}$ exists. Then the tangential and normal components of acceleration are given by

$$a_{\mathbf{T}} = \|\boldsymbol{v}\|' = \boldsymbol{a}\cdot\mathbf{T} = \frac{\boldsymbol{v}\cdot\boldsymbol{a}}{\|\boldsymbol{v}\|}\,,$$
$$a_{\mathbf{N}} = \|\boldsymbol{v}\|\|\mathbf{T}'\| = \boldsymbol{a}\cdot\mathbf{N} = \frac{\|\boldsymbol{v}\times\boldsymbol{a}\|}{\|\boldsymbol{v}\|} = \sqrt{\|\boldsymbol{a}\|^2 - a_{\mathbf{T}}^2}\,.$$

Proof. It suffices to show the formula $a_{\mathbf{N}} = \dfrac{\|\boldsymbol{v}\times\boldsymbol{a}\|}{\|\boldsymbol{v}\|}$. Since $\boldsymbol{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, we find that

$$\boldsymbol{a}\times\mathbf{T} = a_{\mathbf{N}}(\mathbf{N}\times\mathbf{T})\,;$$

thus using the fact that $a_{\mathbf{N}} \geqslant 0$, by Theorem 10.6 we find that

$$a_{\mathbf{N}} = |a_{\mathbf{N}}| = \frac{\|a\times\mathbf{T}\|}{\|\mathbf{N}\times\mathbf{T}\|} = \frac{\|\boldsymbol{a}\times\mathbf{T}\|}{\|\mathbf{N}\|\|\mathbf{T}\|\sin\frac{\pi}{2}} = \|\boldsymbol{a}\times\mathbf{T}\| = \frac{\|\boldsymbol{v}\times\boldsymbol{a}\|}{\|\boldsymbol{v}\|}\,.$$ □