- **Differentiation of determinant functions**

For an $n \times n$ matrix A, let $\mathrm{Cof}(A)$ denote the cofactor matrix of A; that is, the (i,j)-th entry of $\mathrm{Cof}(A)$ is the determinant of the matrix obtained by deleting the i-th row and j-th column of A or

$$\big[\mathrm{Cof}(A)\big]_{ij} = (-1)^{i+j}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1\mathrm{n}} \\ \vdots & \ddots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)\mathrm{n}} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)\mathrm{n}} \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{\mathrm{n}1} & a_{\mathrm{n}2} & \cdots & a_{\mathrm{n}(j-1)} & a_{\mathrm{n}(j+1)} & \cdots & a_{\mathrm{nn}} \end{vmatrix}.$$

Then the determinant of A, using the reductive algorithm, can be computed by

$$\det(A) = \sum_{k=1}^{n} a_{ik}\big[\mathrm{Cof}(A)\big]_{ik} \qquad \forall\, 1 \leqslant i \leqslant n\,. \tag{0.1}$$

On the other hand, the determinant of an $n \times n$ matrix $A = [a_{ij}]_{1\leqslant i,j\leqslant n}$ can be viewed as a real-valued function of n^2 variable:

$$f(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, a_{31}\cdots, a_{nn}) = \det([\mathrm{a_{ij}}])\,.$$

Since for each $1 \leqslant i \leqslant n$ the (i,k)-th entry of the cofactor matrix $\mathrm{Cof}(A)_{ik}$ is independent of a_{ij} for all $1 \leqslant j,k \leqslant n$, we have $\dfrac{\partial f}{\partial a_{ij}} = \big[\mathrm{Cof}(A)\big]_{ij}$; thus if $a_{ij} = a_{ij}(t)$ is a function of t for all $1 \leqslant i,j \leqslant n$, with $A = A(t) = \big[a_{ij}(t)\big]_{1\leqslant i,j\leqslant n}$ in mind the chain rule implies that

$$\frac{d}{dt} f\big(a_{11}(t), a_{12}(t), \cdots, a_{nn}(t)\big) = \sum_{i,j=1}^{n} \big[\mathrm{Cof}(A)\big]_{ij}\frac{da_{ij}(t)}{dt}\,. \tag{0.2}$$

Let $\mathrm{Adj}(A)$ be the transpose of the cofactor matrix, called the adjoint matrix, of A, then (0.2) implies that

$$\frac{d}{dt}\det(A) = \sum_{i,j=1}^{n}\big[\mathrm{Adj}(A)\big]_{ji}\frac{da_{ij}}{dt} = \mathrm{tr}\Big(\mathrm{Adj}(A)\frac{dA}{dt}\Big)\,, \tag{0.3}$$

where $\mathrm{tr}(M)$ denotes the trace of a square matrix M and $\dfrac{dA}{dt} = \Big[\dfrac{da_{ij}}{dt}\Big]_{1\leqslant i,j\leqslant n}$. In particular, if A is invertible, then $A^{-1} = \dfrac{1}{\det(A)}\mathrm{Adj}(A)$; thus for invertible matrix $A = \big[a_{ij}(t)\big]$, we have

$$\frac{d}{dt}\det(A) = \mathrm{tr}\Big(\det(A)A^{-1}\frac{dA}{dt}\Big) = \det(A)\mathrm{tr}\Big(A^{-1}\frac{dA}{dt}\Big) \tag{0.4}$$

or

$$\frac{d}{dt}\ln\big|\det(A)\big| = \mathrm{tr}\Big(A^{-1}\frac{dA}{dt}\Big)\,.$$

Example 0.1. Let $\mathrm{A}(t) = \begin{bmatrix} f(t) & g(t) \\ h(t) & k(t)\end{bmatrix}$. Then

$$\begin{aligned}\frac{d}{dt}\det(\mathrm{A}) &= \mathrm{tr}\Big(\begin{bmatrix} k & -g \\ -h & f\end{bmatrix}\begin{bmatrix} f' & g' \\ h' & k'\end{bmatrix}\Big) = \mathrm{tr}\Big(\begin{bmatrix} kf'-gh' & kg'-gk' \\ -hf'+fh' & -hg'+fk'\end{bmatrix}\Big) \\ &= kf'-gh'-hg'+fk' = (fk-gh)'\,.\end{aligned}$$