Calculus MA1002-B Midterm 3 Sample

Problem 1. 五小題是非題，有倒扣（此大題考觀念，因此對定義或定理需要具備一定的熟悉度方能正確作答，因此請同學對每個定理在什麼條件下會有什麼結果多做複習與了解）

Problem 2. 定義、定理敘述題

Problem 3. Assume that f is a continuous function of two variable satisfying that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - (e^{2x} - 1)\cos y}{x^2 + y^2} = 0\,.$$

Note that the equality above does **NOT** imply that $f(x,y) = (e^{2x} - 1)\cos y$.

1. Find $f_x(0,0)$ and $f_y(0,0)$.

2. Prove or disprove that f is differentiable at $(0,0)$.

Problem 4. Suppose that $u = u(x,y,z)$ is a twice continuously differentiable function of x, y, z satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\,.$$

Let $v(\rho, \theta, \phi) = u(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$. Show that v satisfies that

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 v}{\partial\theta^2} + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial v}{\partial\phi}\right) = 0\,.$$

Problem 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x+y)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Find the direction along which the value of the function f at $(0,0)$ decreases most rapidly.

Problem 6. Find the second Taylor polynomial of the function $f(x,y) = \arcsin\dfrac{y}{\sqrt{x^2 + y^2}}$ at $(1,1)$.

Problem 7. Find all relative extrema and saddle points of $f(x,y) = 2xy - \dfrac{1}{2}(x^4 + y^4) + 1$ using the second derivative test. When a relative extremum is found, determine if it is a relative maximum or a relative minimum.

Problem 8. Find the extrema of the function $f(x,y,z) = x^2 + y^2 + z^2 - z$ on the set

$$R = \left\{(x,y,z) \,\middle|\, (x^2 + 2y^2 - 1)^2 \leqslant 2z^2 \leqslant 4\right\}.$$