Calculus MA1002-B Midterm 3

National Central University, Jun. 09, 2020

Problem 1. (10%) **True** or **False**（是非題）：每題兩分，答對得兩分，答錯倒扣兩分（倒扣至本大題零分為止）

In the following, R is always an open region in the plane, (a,b) is always a point in R, and $f : R \to \mathbb{R}$ is a function of two variables.

F 1. If $\lim_{r\to 0} f(a + r\cos\theta, b + r\sin\theta)$ exists for all $\theta \in \mathbb{R}$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

T 2. If f is differentiable at (a,b), then f_x and f_y both exist (a,b).

T 3. If f_x and f_y are continuous on R, then f is continuous on R.

F 4. If f_x and f_y the directional derivative of f at (a,b) exists in all directions, then f is differentiable at (a,b).

F 5. If f_{xy} and f_{yx} both exist on R, then $f_{xy} = f_{yx}$ on R.

Problem 2. Complete the following.

(1) (5%) Let R be an open region in the plane, $f : R \to \mathbb{R}$ be a function, and $(a,b) \in R$. Define the differentiability of f at (a,b). （定義 f 在 (a,b) 的可微性）

(2) (5%) Let R be an open region in the plane, $f, g : R \to \mathbb{R}$ be differentiable functions of two variables. State the Lagrange Multiplier Theorem (for finding extrema of f subject to constraint $g = 0$). （敘述雙變數函數在一個限制式下的拉格朗日乘子定理）

Problem 3. Assume that f is a continuous function of two variable satisfying that

$$\lim_{(x,y)\to(\pi,1)} \frac{f(x,y) - y\cos x}{(x-\pi)^2 + (y-1)^2} = 0\,.$$

1. (10%) Find $f_x(\pi,1)$ and $f_y(\pi,1)$.

2. (5%) Prove or disprove that f is differentiable at $(\pi,1)$.

Solution. Note that since $\lim_{(x,y)\to(\pi,1)} \frac{f(x,y) - y\cos x}{(x-\pi)^2 + (y-1)^2} = 0$, we must have

$$\lim_{(x,y)\to(\pi,1)} \frac{f(x,y) - y\cos x}{\sqrt{(x-\pi)^2 + (y-1)^2}} = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(\pi,1)} \big[f(x,y) - y\cos x\big] = 0\,.$$

Therefore, $\lim_{(x,y)\to(\pi,1)} f(x,y) = -1$. By the continuity of f, $f(\pi,1) = -1$.

For $(x,y) \neq (\pi,1)$,

$$\frac{f(x,y) - y\cos x}{\sqrt{(x-\pi)^2 + (y-1)^2}} = \frac{f(x,y) - f(\pi,1) + (y-1)}{\sqrt{(x-\pi)^2 + (y-1)^2}} - \frac{y + y\cos x}{\sqrt{(x-\pi)^2 + (y-1)^2}}\,.$$

By Taylor's Theorem, for each x there exists ξ between x and π such that

$$\cos x = \cos\pi - \frac{\cos\xi}{2}(x-\pi)^2 = -1 - \frac{\cos\xi}{2}(x-\pi)^2\,;$$

thus

$$\Big|\frac{y+y\cos x}{\sqrt{(x-\pi)^2+(y-1)^2}}\Big| = \frac{|y||1+\cos x|}{\sqrt{(x-\pi)^2+(y-1)^2}} \leqslant \frac{|y|}{2}\frac{|x-\pi|^2}{\sqrt{(x-\pi)^2+(y-1)^2}} \leqslant \frac{1}{2}|y||x-\pi|^{\frac{3}{2}}$$

and the right-hand side approaches zero as $(x,y)\to(\pi,1)$. By the Squeeze Theorem,

$$\lim_{(x,y)\to(\pi,1)} \frac{y+y\cos x}{\sqrt{(x-\pi)^2+(y-1)^2}} = 0\,;$$

thus

$$\lim_{(x,y)\to(\pi,1)} \frac{\big|f(x,y)-f(\pi,1)+(y-1)\big|}{\sqrt{(x-\pi)^2+(y-1)^2}} = 0\,.$$

The equality above implies that $\underline{f \text{ is differentiable at } (\pi,1)}$ and $\underline{f_x(\pi,1)=0}$, $\underline{f_y(\pi,1)=-1}$. □

Problem 4. (12%) Suppose that $c_1, c_2 \in \mathbb{R}$ are constants, and $u = u(x,y,t)$ is a twice differentiable function of x, y, t satisfying $u_{xy} = u_{yx}$ and

$$\frac{\partial u}{\partial t} + c_1\frac{\partial u}{\partial x} + c_2\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,.$$

Let $v(r,\theta,t) = u(r\cos\theta + c_1 t, r\sin\theta + c_2 t, t)$. Show that v satisfies that

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\,.$$

Proof. Since $v(r,\theta,t) = u(r\cos\theta + c_1 t, r\sin\theta + c_2 t, t)$, by the chain rule

$$\begin{aligned} v_t &= u_x c_1 + u_y c_2 + u_t\,, \\ v_r &= u_x\cos\theta + u_y\sin\theta\,, \\ v_\theta &= u_x r(-\sin\theta) + u_y r\cos\theta = -u_x r\sin\theta + u_y r\cos\theta\,; \end{aligned}$$

thus by the fact that $u_{xy} = u_{yx}$ we have

$$\begin{aligned} v_{rr} &= u_{xx}\cos^2\theta + u_{xy}\cos\theta\sin\theta + u_{yx}\sin\theta\cos\theta + u_{yy}\sin^2\theta \\ &= u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta\,, \end{aligned}$$

and

$$\begin{aligned} v_{\theta\theta} &= u_{xx}r^2\sin^2\theta - u_{xy}r^2\sin\theta\cos\theta - u_x r\cos\theta - u_{yx}r^2\sin\theta\cos\theta + u_{yy}r^2\cos^2\theta - u_y r\sin\theta \\ &= u_{xx}r^2\sin^2\theta - 2u_{xy}r^2\sin\theta\cos\theta + u_{yy}r^2\cos^2\theta - u_x r\cos\theta - u_y r\sin\theta\,. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \\ &= u_t + c_1 u_x + c_2 u_y - u_{xx}\cos^2\theta - 2u_{xy}\sin\theta\cos\theta - u_{yy}\sin^2\theta - \frac{1}{r}\big(u_x\cos\theta + u_y\sin\theta\big) \\ &\quad - \frac{1}{r^2}\Big(u_{xx}r^2\sin^2\theta - 2u_{xy}r^2\sin\theta\cos\theta + u_{yy}r^2\cos^2\theta - u_x r\cos\theta - u_y r\sin\theta\Big) \\ &= u_t + c_1 u_x + c_2 u_y - u_{xx} - u_{yy} = 0 \end{aligned}$$

which shows $\dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2}$. □

Problem 5. (8%) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{y^4(3x+4y)}{x^6+5y^4} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Find the direction along which the value of the function f at $(0,0)$ increases most rapidly.
(找出在 $(0,0)$ 點 f 的函數值上升最快的方向)

Proof. Let $\boldsymbol{u} = (\cos\theta, \sin\theta)$. Then

$$\begin{aligned}(D_{\boldsymbol{u}}f)(0,0) &= \lim_{t\to 0}\frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} = \lim_{t\to 0}\frac{t^4\sin^4\theta(3t\cos\theta + 4t\sin\theta)}{t(t^6\cos^6\theta + 5t^4\sin^4\theta)} \\ &= \begin{cases} 0 & \text{if } \sin\theta = 0\,, \\ \dfrac{3\cos\theta + 4\sin\theta}{5} & \text{if } \sin\theta \neq 0\,. \end{cases}\end{aligned}$$

The direction along which the value of f at $(0,0)$ increases most rapidly is the direction which maximize $(D_{\boldsymbol{u}}f)(0,0)$. Since the maximum of $(D_{\boldsymbol{u}}f)(0,0)$ occurs at $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$, the direction along which the value of f at $(0,0)$ increases most rapidly is $\big(\frac{3}{5}, \frac{4}{5}\big)$. □

Problem 6. (12%) Find the second Taylor polynomial of the function $f(x,y) = \arctan(y\tan x)$ at $\big(\frac{3\pi}{4}, 1\big)$.

Solution. By the chain rule implies that

$$\begin{aligned} f_x(x,y) &= \frac{y\sec^2 x}{1+y^2\tan^2 x}\,, \qquad f_y(x,y) = \frac{\tan x}{1+y^2\tan^2 x}\,, \\ f_{xx}(x,y) &= \frac{2y\sec^2 x\tan x\cdot(1+y^2\tan^2 x) - 2y^2\sec^2 x\tan x\cdot y\sec^2 x}{(1+y^2\tan^2 x)^2}\,, \\ f_{xy}(x,y) &= \frac{\sec^2 x\cdot(1+y^2\tan^2 x) - 2y\tan^2 x\cdot y\sec^2 x}{(1+y^2\tan^2 x)^2}\,, \quad f_{yy}(x,y) = \frac{-2y\tan^2 x\cdot\tan x}{(1+y^2\tan^2 x)^2}\,; \end{aligned}$$

thus using that $\tan\frac{3\pi}{4} = -1$ and $\sec\frac{3\pi}{4} = -\sqrt{2}$, we find that

$$\begin{aligned} &f_x\big(\tfrac{3\pi}{4},1\big) = 1\,, \quad f_y\big(\tfrac{3\pi}{4},1\big) = -\frac{1}{2}\,, \quad f_{xx}\big(\tfrac{3\pi}{4},1\big) = \frac{-8+8}{4} = 0\,, \\ &f_{xy}\big(\tfrac{3\pi}{4},1\big) = \frac{4-4}{4} = 0\,, \quad f_{yy}\big(\tfrac{3\pi}{4},1\big) = \frac{2}{4} = \frac{1}{2}\,. \end{aligned}$$

Since $f\big(\frac{3\pi}{4},1\big) = \arctan(\tan\frac{3\pi}{4}) = \arctan(-1) = -\frac{\pi}{4}$, we find that the second Taylor polynomial of f at $\big(\frac{3\pi}{4},1\big)$ is

$$\begin{aligned} \underline{P_2(x,y)} &= f\big(\tfrac{3\pi}{4},1\big) + f_x\big(\tfrac{3\pi}{4},1\big)\big(x-\tfrac{3\pi}{4}\big) + f_y\big(\tfrac{3\pi}{4},1\big)(y-1) \\ &\quad + \frac{1}{2!}\Big[f_{xx}\big(\tfrac{3\pi}{4},1\big)\big(x-\tfrac{3\pi}{4}\big)^2 + 2f_{xy}\big(\tfrac{3\pi}{4},1\big)\big(x-\tfrac{3\pi}{4}\big)(y-1) + f_{yy}\big(\tfrac{3\pi}{4},1\big)(y-1)^2\Big] \\ &= -\frac{\pi}{4} + \big(x-\tfrac{3\pi}{4}\big) - \frac{1}{2}(y-1) + \frac{1}{4}(y-1)^2\,. \end{aligned}$$

□

Problem 7. (13%) Let $k > 1$ be a real number. Find all relative extrema and saddle points of $f(x,y) = (x^2 + ky^2)e^{-x^2-y^2}$ using the second derivative test. When a relative extremum is found, determine if it is a relative maximum or a relative minimum.

Solution. We first compute the first and second partial derivatives of f and find that

$$\begin{aligned}
f_x(x,y) &= 2xe^{-x^2-y^2} + (x^2+ky^2)(-2x)e^{-x^2-y^2} = 2x(1-x^2-ky^2)e^{-x^2-y^2}\,,\\
f_y(x,y) &= 2kye^{-x^2-y^2} + (x^2+ky^2)(-2y)e^{y^2-x^2} = 2y(k-x^2-ky^2)e^{-x^2-y^2}\,,\\
f_{xx}(x,y) &= \big[2-6x^2-2ky^2-4x^2(1-x^2-ky^2)\big]e^{-x^2-y^2}\,,\\
f_{xy}(x,y) &= \big[2x(-2ky)-4xy(1-x^2-ky^2)\big]e^{-x^2-y^2}\,,\\
f_{yy}(x,y) &= \big[2k-2x^2-6ky^2-4y^2(k-x^2-ky^2)\big]e^{-x^2-y^2}\,.
\end{aligned}$$

Therefore, critical points of f are $(0,0)$, $(\pm 1,0)$ and $(0,\pm 1)$.

1. Since $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 2k$, $f_{xy}(0,0) = 0$, we find that
$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 4k > 0\,;$$
thus the fact that $f_{xx}(0,0) > 0$ implies that $\underline{f(0,0) \text{ is a relative minimum of } f}$.

2. Since $f_{xx}(\pm 1,0) = -4e^{-1}$, $f_{yy}(1,0) = 2(k-1)e^{-1}$ and $f_{xy}(1,0) = 0$, we find that
$$f_{xx}(\pm 1,0)f_{yy}(\pm 1,0) - f_{xy}(\pm 1,0)^2 = -8(k-1)e^{-2} < 0\,;$$
thus $\underline{(\pm 1,0) \text{ is a saddle point of } f}$.

3. Since $f_{xx}(0,\pm 1) = 2(1-k)e^{-1}$, $f_{yy}(0,\pm 1) = -4ke^{-1}$ and $f_{xy}(0,\pm 1) = 0$, we find that
$$f_{xx}(0,\pm 1)f_{yy}(0,\pm 1) - f_{xy}(0,\pm 1)^2 = 8k(k-1)e^{-2} > 0\,;$$
thus the fact that $f_{xx}(0,\pm) < 0$ implies that $\underline{f(0,\pm 1) \text{ is a relative maximum of } f}$. □

Problem 8. (20%) Find the extreme value of the function $f(x,y,z) = 2x^2 + 2y^2 + 2z^2 - z$ on the set
$$R = \big\{(x,y,z)\,\big|\,(2x^2+y^2-1)^2 \leqslant z^2 \leqslant 4\big\}\,.$$

Solution. Suppose that f attains its maximum at $(x_0,y_0,z_0) \in R$.

1. If (x_0,y_0,z_0) is an interior point of R, then
$$(\nabla f)(x_0,y_0,z_0) = (4x_0, 4y_0, 4z_0-1) = \mathbf{0}$$
which implies that $(x_0,y_0,z_0) = \big(0,0,\frac{1}{4}\big)$. This point does not belong to R; thus f does not attain its extreme value in the interior of R.

2. Suppose that (x_0,y_0,z_0) on the boundary $z^2 = 4$. Then $z_0 = \pm 2$, and $f(x_0,y_0,2) = 2x^2+2y^2+6$, $f(x_0,y_0,-2) = 2x^2+2y^2+10$ whose minimum is 6.

3. Suppose that (x_0, y_0, z_0) on the boundary $(2x^2+y^2-1)^2 = z^2$. Let $g(x,y,z) = (2x^2+y^2-1)^2 - z^2$. Then

$$(\nabla g)(x,y,z) = \big(8x(2x^2+y^2-1), 4y(2x^2+y^2-1), -2z\big).$$

(a) If $(\nabla g)(x_0,y_0,z_0) = \mathbf{0}$, then $z_0 = 0$ and $2x_0^2+y_0^2 = 1$. Subject to the constraint $2x_0^2+y_0^2 = 1$, $f(x_0,y_0,z_0) = 2x_0^2+2y_0^2$ attains its maximum at $(x_0,y_0) = (0,\pm 1)$ with value 2 and attains its minimum at $(x_0,y_0) = \big(\pm\frac{1}{\sqrt{2}}, 0\big)$ with value 1.

(b) If $(\nabla g)(x_0,y_0,z_0) \neq \mathbf{0}$, then there exists $\lambda \in \mathbb{R}$ such that

$$(4x_0, 4y_0, 4z_0 - 1) = \lambda\big(8x_0(2x_0^2+y_0^2-1), 4y_0(2x_0^2+y_0^2-1), -2z_0\big).$$

which implies that

$$x_0 = 2\lambda x_0(2x_0^2+y_0^2-1), \tag{0.1a}$$
$$y_0 = \lambda y_0(2x_0^2+y_0^2-1), \tag{0.1b}$$
$$4z_0 - 1 = -2\lambda z_0, \tag{0.1c}$$
$$z_0^2 = (2x_0^2+y_0^2-1)^2. \tag{0.1d}$$

Note that (0.1c) implies that $z_0 \neq 0$ and (0.1d) implies that $\lambda \neq 0$ (for otherwise we must have $x_0 = y_0 = 0$ and $z_0 = \frac{1}{4}$ that do not satisfy (0.1d)).

i. If $(x_0,y_0) = (0,0)$, then $z_0 = \pm 1$ and we have $f(0,0,1) = 1$ and $f(0,0,-1) = 3$.

ii. If $x_0 \neq 0$, then $2x_0^2 + y_0^2 - 1 = \frac{1}{2\lambda}$; thus (0.1b) implies that $y_0 = 0$. Therefore, $2x_0^2 - 1 = \frac{1}{2\lambda}$ and (0.1c) shows that $z_0 = \frac{1}{2\lambda+4}$. Therefore, using (0.1d) we find that

$$\frac{1}{4\lambda^2} = \frac{1}{(2\lambda+4)^4} = \frac{1}{4(\lambda+2)^2};$$

thus $\lambda^2 = (\lambda+2)^2$ which shows that $\lambda = -1$. Therefore, $x_0^2 = \frac{1}{4}$ and $z_0 = \frac{1}{2}$ so that

$$f(x_0,y_0,z_0) = 2\cdot\frac{1}{4} + 2\cdot 0 + 2\cdot\frac{1}{4} - \frac{1}{2} = \frac{1}{2}.$$

iii. If $y_0 \neq 0$, then $2x_0^2 + y_0^2 - 1 = \frac{1}{\lambda}$ so that (0.1a) implies that $x_0 = 0$. Therefore, $y_0^2 = 1 + \frac{1}{\lambda}$. Together with the fact that $z_0 = \frac{1}{2\lambda+4}$, we find that

$$\frac{1}{\lambda^2} = \frac{1}{(2\lambda+4)^2} = \frac{1}{4(\lambda+2)^2}.$$

Therefore, $\lambda = -4$ or $\lambda = -\frac{4}{3}$.

A. If $\lambda = -4$, then $y_0^2 = \frac{3}{4}$ and $z_0 = -\frac{1}{4}$. In this case,

$$f(x_0,y_0,z_0) = 2\cdot 0 + 2\cdot\frac{3}{4} + 2\cdot\frac{1}{16} + \frac{1}{4} = \frac{15}{8}.$$

B. If $\lambda = -\frac{4}{3}$, then $y_0^2 = \frac{1}{4}$ and $z_0 = \frac{3}{4}$. In this case,

$$f(x_0, y_0, z_0) = 2 \cdot 0 + 2 \cdot \frac{1}{4} + 2 \cdot \frac{9}{16} - \frac{3}{4} = \frac{7}{8}.$$

iv. If $x_0, y_0 \neq 0$, then (0.1a,b) implies that $2x_0^2 + y_0^2 - 1 = 0$ which further implies that $z_0 = 0$, a contradiction.

4. Suppose that (x_0, y_0, z_0) satisfies both $z^2 = 4$ and $(2x^2 + y^2 - 1)^2 = z^2$. Then $2x_0^2 + y_0^2 - 1 = 2$; thus $2x_0^2 + y_0^2 = 3$. In this case, $f(x_0, y_0, 2)$ attaints its maximum at $(0, \pm\sqrt{3}, 2)$ with value 12, while $f(x_0, y_0, -2)$ attains its maximum at $(0, \pm\sqrt{3}, -2)$ with value 16.

Comparing all the possible extrema, we find that the minimum of f on R is $\frac{1}{2}$ and the maximum of f on R is 16. □