Calculus MA1002-B Midterm 2

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Problem 1. Complete the following.

1. (5%) State the limit comparison test for the convergence or divergence of infinite series.
2. (15%) State and prove the Taylor Theorem (you can directly apply the Cauchy Mean Value Theorem without stating it).

Solution. 1. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, $a_n, b_n > 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L\,,$$

where L is a non-zero real number. Then $\sum\limits_{k=1}^{\infty} a_k$ converges if and only if $\sum\limits_{k=1}^{\infty} b_k$ converges.

2. Let $f : (a,b) \to \mathbb{R}$ be $(n+1)$-times differentiable, and $c \in (a,b)$. Then for each $x \in (a,b)$, there exists ξ between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\,. \quad (\star)$$

Proof. We first show that if $h : (a,b) \to \mathbb{R}$ is m-times differentiable, and $c \in (a,b)$. Then for all $d \in (a,b)$ and $d \neq c$ there exists ξ between c and d such that

$$\frac{h(d) - \sum\limits_{k=0}^{m} \frac{h^{(k)}(c)}{k!}(d-c)^k}{(d-c)^{m+1}} = \frac{1}{m+1} \frac{h'(\xi) - \sum\limits_{k=0}^{m-1} \frac{(h')^{(k)}(c)}{k!}(\xi-c)^k}{(\xi-c)^m}\,. \quad (\square)$$

Let $F(x) = h(x) - \sum\limits_{k=0}^{m} \frac{h^{(k)}(c)}{k!}(x-c)^k$ and $G(x) = (x-c)^m$. Then F, G are continuous on $[c,d]$ (or $[d,c]$) and differentiable on (c,d) (or (d,c)), and $G'(x) \neq 0$ for all $x \neq c$. Therefore, the Cauchy Mean Value Theorem implies that there exists ξ between c and d such that

$$\frac{F(d) - F(c)}{G(d) - G(c)} = \frac{F'(\xi)}{G'(\xi)}\,,$$

and $(\square)$ is exactly the explicit form of the equality above.

Now we apply $(\square)$ successfully for $h = f,\ f',\ f'',\ \cdots$ and $f^{(n)}$ and find that

$$\begin{aligned}
\frac{f(d) - \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(d-c)^k}{(d-c)^{n+1}} &= \frac{1}{n+1} \frac{f'(d_1) - \sum\limits_{k=0}^{n-1} \frac{(f')^{(k)}(c)}{k!}(d_1-c)^k}{(d_1-c)^n} \\
&= \frac{1}{n+1} \cdot \frac{1}{n} \frac{f''(d_2) - \sum\limits_{k=0}^{n-2} \frac{(f'')^{(k)}(c)}{k!}(d_2-c)^k}{(d_2-c)^{n-1}} \\
&= \cdots\cdots \\
&= \frac{1}{(n+1)!} \frac{f^{(n)}(d_n) - f^{(n)}(c)}{d_n - c} = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\,;
\end{aligned}$$

thus

$$f(d) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(d-c)^k = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(d-c)^{n+1} \,.$$

($\star$) then follows from the equality above since $d \in (a,b)$ is given arbitrary. □

Problem 2. (10%) Use the ratio test to determine the whether the series $\sum_{n=0}^{\infty} \frac{n!\, n^n}{(2n+1)!}$ converges or not.

Solution. Let $a_n = \frac{n!\, n^n}{(2n+1)!}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{(n+1)!\,(n+1)^{n+1}}{(2(n+1)+1)!}}{\dfrac{n!\,n^n}{(2n+1)!}} = \frac{(n+1)(n+1)^{n+1}}{(2n+3)(2n+2)n^n} = \frac{n+1}{2(2n+3)}\Big(1+\frac{1}{n}\Big)^n ;$$

thus

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \Big[\frac{n+1}{2(2n+3)}\Big(1+\frac{1}{n}\Big)^n\Big] = \frac{e}{4} \,.$$

Since $\frac{e}{4} < 1$, the ratio test implies that the series $\sum_{n=0}^{\infty} \frac{n!\, n^n}{(2n+1)!}$ converges (absolutely). □

Problem 3. (15%) Find all $p \in \mathbb{R}$ such that $\sum_{k=2}^{\infty} \frac{\exp\big(\frac{1}{k}\big) - 1}{(\ln k)^p}$ converges. Note that you need to provide the reason for the convergence or divergence of the power series for each p.

Proof. Let $a_k = \frac{\exp\big(\frac{1}{k}\big) - 1}{(\ln k)^p}$ and $b_k = \frac{1}{k(\ln k)^p}$. Then $a_k, b_k > 0$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{e^{\frac{1}{k}} - 1}{\frac{1}{k}} = 1$$

since $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. By the limit comparison test, $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. Since $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges if and only if $p > 1$, we find that $\sum_{k=2}^{\infty} \frac{\exp\big(\frac{1}{k}\big) - 1}{(\ln k)^p}$ converges if and only $p > 1$. □

Problem 4. (15%) Show that $\sum_{k=1}^{\infty} \frac{(-1)^k \sin(kx)}{k}$ converges for all $x \in \mathbb{R}$.

Proof. First we note that since the function $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \sin kx}{k}$, when convergent, is 2π periodic; thus we only need to show that the series converges for $x \in [-\pi, \pi]$. Since $f(\pi) = f(-\pi) = 0$, it suffices to show that the series converges for all $x \in (-\pi, \pi)$. Note that $(-1)^k \sin kx = \sin(kx + k\pi) =$

$\sin k(x+\pi)$; thus

$$\begin{aligned}
2\sin\frac{x+\pi}{2}\sum_{k=1}^{n}(-1)^k\sin kx &= \sum_{k=1}^{n} 2\sin\frac{x+\pi}{2}\sin k(x+\pi)\\
&= \sum_{k=1}^{n}\cos\big[\big(k-\frac{1}{2}\big)(x+\pi)\big]-\cos\big[\big(k+\frac{1}{2}\big)(x+\pi)\big]\\
&= \cos\frac{x+\pi}{2}-\cos\frac{3(x+\pi)}{2}+\cos\frac{3(x+\pi)}{2}-\cos\frac{5(x+\pi)}{2}+\cdots+\cos\frac{(2n-1)(x+\pi)}{2}\\
&\quad -\cos\frac{(2n+1)(x+\pi)}{2}\\
&= \cos\frac{x+\pi}{2}-\cos\frac{(2n+1)(x+\pi)}{2}\,.
\end{aligned}$$

Therefore, by the fact that $\sin\frac{x+\pi}{2}\neq 0$ if $x\in(-\pi,\pi)$, we find that

$$\sum_{k=1}^{n}(-1)^k\sin kx=\frac{1}{2\sin\frac{x+\pi}{2}}\Big[\cos\frac{x+\pi}{2}-\cos\frac{(2n+1)(x+\pi)}{2}\Big]\,;$$

thus

$$\Big|\sum_{k=1}^{n}(-1)^k\sin kx\Big|\leqslant\frac{1}{\sin\frac{x+\pi}{2}}\qquad\forall\,x\in(-\pi,\pi)\,.$$

Since the right-hand side is independent of n, by the Dirichlet test we find that

$$\sum_{k=1}^{\infty}\frac{(-1)^k\sin kx}{k}\text{ converges for all }x\in(-\pi,\pi).$$

By the argument above, we conclude that the series $\sum\limits_{k=1}^{\infty}\frac{(-1)^k\sin kx}{k}$ converges for all $x\in\mathbb{R}$. □

Problem 5. (10%) Let k be a natural number. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{(n!)^k}{(kn)!}x^{2n}\,.$$

If you are not able to complete this question with general natural number k, try to complete the case $k=3$.

Solution. Let $k\in\mathbb{N}$ be given, and $a_n=\frac{(n!)^k}{(kn)!}$. Then $a_n>0$ and

$$\frac{a_n}{a_{n+1}}=\frac{\frac{(n!)^k}{(kn)!}}{\frac{[(n+1)!]^k}{(kn+k)!}}=\frac{(kn+k)(kn+k-1)\cdots(kn+1)}{(n+1)^k}\,.$$

Therefore,

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=k^k\,;$$

thus the radius of convergence of the given series is $k^{\frac{k}{2}}$. Now we check the convergence or divergence of the series at the end-points $\pm k^{\frac{k}{2}}$: $\sum\limits_{n=0}^{\infty} \dfrac{(n!)^k k^{kn}}{(kn)!}$. Using the Stirling formula and the limit comparison test, it suffices to consider the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(\sqrt{2\pi n} n^n e^{-n})^k k^{kn}}{\sqrt{2\pi kn}(kn)^{kn} e^{-kn}} = \sum_{n=1}^{\infty} \frac{\sqrt{2\pi n}^k n^{kn} e^{-kn} k^{kn}}{\sqrt{2\pi kn}(kn)^{kn} e^{-kn}} = \sum_{n=1}^{\infty} \sqrt{2\pi}^{k-1} n^{\frac{k-1}{2}} .$$

If $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} n^{\frac{k-1}{2}} \neq 0$; thus the n-th term's test implies that the series does not converge at the end-points. Therefore, the interval of convergence is $(-k^{\frac{k}{2}}, k^{\frac{k}{2}})$. □

Problem 6. (10%) Find the power series solution $y(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ of the differential equation

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0 , \qquad y(0) = 1 , \quad y'(0) = 0 .$$

You need to show all the computations instead of just providing the answer.

Solution. First we note that $y(0) = 1$ implies that $a_0 = 1$ and $y'(0) = 0$ implies that $a_1 = 0$. By the differentiation of power series, we find that

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Therefore, by the fact that $a_1 = 0$, we have

$$\sum_{n=2}^{\infty} \big[n(n-1)a_n + n a_n + a_{n-2}\big] x^n = 0$$

which implies that $n^2 a_n + a_{n-2} = 0$ for all $n \geqslant 2$, or equivalently,

$$a_n = \frac{-1}{n^2} a_{n-2} \qquad \forall\, n \geqslant 2 .$$

Since $a_1 = 0$, we must have $a_1 = a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0$. Moreover, since $a_0 = 1$, we find that

$$a_{2n} = \frac{-1}{(2n)^2} a_{2n-2} = \frac{(-1)^2}{(2n)^2(2n-2)^2} a_{2n-4} = \cdots = \frac{(-1)^n}{(2n)^2(2n-2)^2 \cdots 2^2} a_0 = \frac{(-1)^n}{2^{2n}(n!)^2} .$$

Therefore, the power series solution of the differential equation is given by

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n} .$$

□

Problem 7. (10%) Find the fourth Maclaurin polynomial of the function $f(x) = \ln \cos x$.

Solution. By the chain rule and the product rule,

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x , \qquad f''(x) = -\sec^2 x ,$$
$$f'''(x) = -2 \sec x \frac{d}{dx} \sec x = -2 \sec^2 x \tan x ,$$
$$f^{(4)}(x) = -2\Big(\frac{d}{dx} \sec^2 x\Big) \tan x - 2 \sec^2 x \frac{d}{dx} \tan x = -4 \sec^2 x \tan^2 x - 2 \sec^4 x .$$

Therefore,

$$f(0) = 0\,, \quad f'(0) = 0\,, \quad f''(0) = -1\,, \quad f^{(3)}(0) = 0\,, \quad f^{(4)}(0) = -2\,,$$

and the fourth Maclaurin series of f is $P_4(x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4$. ◻

Problem 8. (10%) Find a natural number n such that

$$\Big| \ln 1.1 - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k \cdot 10^k} \Big| < 2 \times 10^{-8}\,.$$

Explain your answer.

Solution. Since

$$\frac{d^k}{dx^k} \ln(1+x) = (-1)^{k-1}(k-1)!(1+x)^{-k} \qquad \forall\, k \in \mathbb{N}\,,$$

the Taylor Theorem implies that for each x there exists ξ between x and 0 such that

$$\begin{aligned} \ln(1+x) &= \sum_{k=1}^{n} \frac{1}{k!} \Big[\frac{d^k}{dx^k}\Big|_{x=0} \ln(1+x) \Big] x^k + \frac{(-1)^n n!(1+\xi)^{-(n+1)}}{(n+1)!} x^{n+1} \\ &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k + \frac{(-1)^n}{n+1} \frac{x^n}{(1+\xi)^n}\,. \end{aligned}$$

In particular, there exists ξ between 0 and $\frac{1}{10}$ such that

$$\ln 1.1 = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k10^k} + \frac{(-1)^n}{n+1} \frac{1}{(1+\xi)^n} 10^{-(n+1)}\,.$$

Therefore, if $n = 7$,

$$\Big| \ln 1.1 - \sum_{k=1}^{7} \frac{(-1)^{k-1}}{k10^k} \Big| \leqslant \frac{1}{8(1+\xi)^8} 10^{-8} < 1.25 \times 10^{-9} < 2 \times 10^{-8}\,.$$ ◻