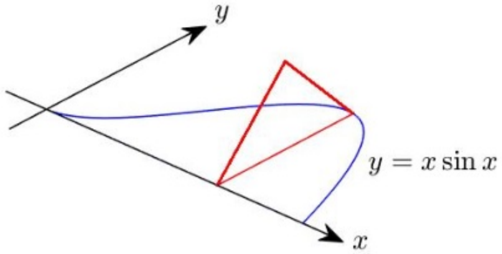
# Calculus MA1002-A Midterm 1

National Central University, Mar. 17, 2019

**Problem 1.** (15%) Find the volume of the solid whose base is the region between the curve $y = x\sin x$ and the interval $[0,\pi]$ on the $x$-axis and the cross-sections perpendicular to the $x$-axis are equilateral triangles（正三角形）with bases running from the $x$-axis to the curve.




*Solution.* Using the method of cross section, the volume of the solid given above is

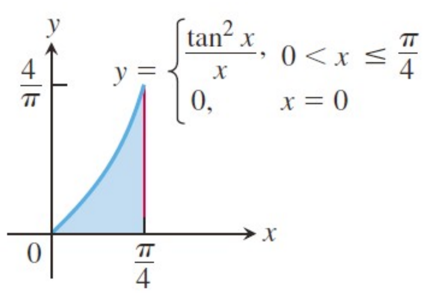
$$\int_0^\pi \frac{\sqrt{3}}{4}x^2\sin^2 x\,dx = \frac{\sqrt{3}}{4}\int_0^\pi x^2\cdot\frac{1-\cos(2x)}{2}\,dx = \frac{\sqrt{3}}{8}\int_0^\pi \big[x^2 - x^2\cos(2x)\big]\,dx$$
$$= \frac{\sqrt{3}}{8}\Big[\frac{\pi^3}{3} - \int_0^\pi x^2\cos(2x)\,dx\Big].$$

Integrating by parts,

$$\int_0^\pi x^2\cos(2x)\,dx = \frac{x^2\sin(2x)}{2}\Big|_{x=0}^{x=\pi} - \int_0^\pi \frac{\sin(2x)}{2}\cdot 2x\,dx = -\int_0^\pi x\sin(2x)\,dx$$
$$= -\Big[-\frac{x\cos(2x)}{2}\Big|_{x=0}^{x=\pi} + \int_0^\pi \frac{\cos(2x)}{2}\,dx\Big] = \frac{\pi}{2}.$$

Therefore, the volume of the given solid is $\frac{\sqrt{3}}{8}\big(\frac{\pi^3}{3} - \frac{\pi}{2}\big)$. □

**Problem 2.** (30%) Find the volume of the solid formed by revolving the shaded region about the $y$-axis shown in the following figure **using at least two different methods**.




*Solution.* Using the shell method, the volume of the given solid is

$$\int_0^{\frac{\pi}{4}} 2\pi x\cdot\frac{\tan^2 x}{x}\,dx = 2\pi\int_0^{\frac{\pi}{4}}\tan^2 x\,dx = 2\pi\int_0^{\frac{\pi}{4}}(\sec^2 x - 1)\,dx = 2\pi(\tan x - x)\Big|_{x=0}^{x=\frac{\pi}{4}} = 2\pi\big(1-\frac{\pi}{4}\big).$$

On the other hand, since the given solid is the complement of a cylinder and a bullet head like solid, the volume of the given solid can be computed by

$$\pi\left(\frac{\pi}{4}\right)^2 \cdot \frac{4}{\pi} - V\,,$$

where $V$ is the volume of the bullet head like solid. Using the disk method,

$$V = \int_0^{\frac{4}{\pi}} \pi\left[f^{-1}(y)\right]^2 dy\,,$$

where $f(x) = \dfrac{\tan^2 x}{x}$. Let $y = f(x)$. Then

$$dy = \frac{2x\tan x\sec^2 x - \tan^2 x}{x^2}\,dx = \frac{2x\tan x\sec^2 x - \tan^2 x}{x^2}\,dx\,;$$

thus the substitution of variable implies that

$$\begin{aligned}
V &= \int_0^{\frac{\pi}{4}} \pi\left[f^{-1}\big(f(x)\big)\right]^2 \frac{2x\tan x\sec^2 x - \tan^2 x}{x^2}\,dx = \pi\int_0^{\frac{\pi}{4}} (2x\tan x\sec^2 x - \tan^2 x)\,dx \\
&= \pi\left[\int_0^{\frac{\pi}{4}} x d(\tan^2 x) - \int_0^{\frac{\pi}{4}} \tan^2 x\,dx\right] = \pi\left[x\tan^2 x\Big|_{x=0}^{x=\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}} \tan^2 x\,dx\right] \\
&= \frac{\pi^2}{4} - 2\pi\int_0^{\frac{\pi}{4}} \tan^2 x\,dx\,.
\end{aligned}$$

Therefore, the volume of the given solid is

$$\frac{\pi^2}{4} - \frac{\pi^2}{4} + 2\pi\left(1 - \frac{\pi}{4}\right) = 2\pi\left(1 - \frac{\pi}{4}\right).$$

□

**Problem 3.** Let $G$ be the graph of the function $y = \sqrt{x - x^2} + \arcsin\sqrt{x}$ on $[0, 1]$.

1. (15%) Find the arc-length of $G$.
2. (15%) Find the area of the surface formed by revolving $G$ about the $x$-axis.

*Solution.* First we compute $y'$ as follows: by the chain rule we obtain that

$$y' = \frac{1}{2\sqrt{x - x^2}} \cdot \frac{d}{dx}(x - x^2) + \frac{1}{\sqrt{1 - \sqrt{x}^2}} \cdot \frac{d}{dx}\sqrt{x} = \frac{1 - 2x}{2\sqrt{x - x^2}} + \frac{1}{\sqrt{1 - x}}\frac{1}{2\sqrt{x}} = \frac{\sqrt{1 - x}}{\sqrt{x}}\,.$$

Therefore, the arc length of the graph is given by

$$\int_0^1 \sqrt{1 + y'^2}\,dx = \int_0^1 \sqrt{1 + \frac{1 - x}{x}}\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_{x=0}^{x=1} = 2\,.$$

Let $S$ be the surface formed by revolving $G$ about the $x$-axis. Then the area of $S$ is given by

$$\begin{aligned}
&\int_0^1 2\pi\frac{\sqrt{x - x^2} + \arcsin\sqrt{x}}{\sqrt{x}}\,dx = 2\pi\int_0^1 \left[\sqrt{1 - x} + \frac{\arcsin\sqrt{x}}{\sqrt{x}}\right] dx \\
&\quad = 2\pi\left[-\frac{2}{3}(1 - x)^{\frac{3}{2}}\Big|_{x=0}^{x=1} + \int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x}}\,dx\right] = \frac{4\pi}{3} + 2\pi\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x}}\,dx\,.
\end{aligned}$$

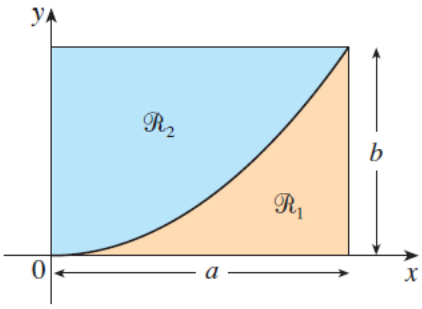
Let $\sqrt{x} = \sin u$. Then $x = \sin^2 u$ which shows that $dx = 2\sin u \cos u\, du$; thus the substitution of variables implies that

$$\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{x}}\,dx = \int_0^{\frac{\pi}{2}} \frac{u}{\sin u}\cdot 2\sin u\cos u\,du = 2\int_0^{\frac{\pi}{2}} u\cos u\,du$$
$$= 2\Big[u\sin u\Big|_{u=0}^{u=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin u\,du\Big] = 2\Big(\frac{\pi}{2}-1\Big) = \pi - 2\,.$$

Therefore, the area of the surface of revolution is

$$\frac{4\pi}{3} + 2\pi(\pi-2) = \frac{4\pi}{3} + 2\pi^2 - 4\pi = 2\pi^2 - \frac{8\pi}{3}\,.$$ □

**Problem 4.** (25%) A rectangle $\mathscr{R}$ with sides $a$ and $b$ is divided into two parts $\mathscr{R}_1$ and $\mathscr{R}_2$ by an arc of a parabola that has its vertex at one corner of $\mathscr{R}$ and passes through the opposite corner. Find the centroids of both $\mathscr{R}_1$ and $\mathscr{R}_2$.




*Solution.* The parabola with vertex at $(0,0)$ and passing through $(a,b)$ is $y = f(x) = \frac{b}{a^2}x^2$. Then the centroid of $\mathscr{R}_1$ is given by

$$(\bar{x}_1,\bar{y}_1) = \frac{1}{\text{Area of }\mathscr{R}_1}\Big(\int_0^a xf(x)\,dx, \frac{1}{2}\int_0^a f(x)^2\,dx\Big) = \frac{1}{\int_0^a \frac{b}{a^2}x^2\,dx}\Big(\int_0^a \frac{b}{a^2}x^3\,dx, \frac{1}{2}\int_0^a \frac{b^2}{a^4}x^4\,dx\Big)$$
$$= \frac{1}{\frac{ab}{3}}\Big(\frac{a^2b}{4},\frac{ab^2}{10}\Big) = \Big(\frac{3a}{4},\frac{3b}{10}\Big)$$

and the centroid of $\mathscr{R}_2$ is given by

$$(\bar{x}_2,\bar{y}_2) = \frac{1}{\text{Area of }\mathscr{R}_2}\Big(\int_0^a x\big[b-f(x)\big]\,dx, \frac{1}{2}\int_0^a \big[b^2 - f(x)^2\big]\,dx\Big)$$
$$= \frac{1}{\int_0^a \big[b-\frac{b}{a^2}x^2\big]\,dx}\Big(\int_0^a \big[b - \frac{b}{a^2}x^2\big]\,dx, \frac{1}{2}\int_0^a \big[b^2 - \frac{b^2}{a^4}x^4\big]\,dx\Big)$$
$$= \frac{1}{\frac{2ab}{3}}\Big(ab - \frac{3ab}{4}, \frac{1}{2}\big(ab^2 - \frac{1}{5}ab^2\big)\Big) = \Big(\frac{3a}{8},\frac{3b}{5}\Big)\,.$$

Note that $(x_1,y_1)$ and $(x_2,y_2)$ satisfy that

$$\frac{\text{Area of }\mathscr{R}_1\cdot(x_1,y_1) + \text{Area of }\mathscr{R}_1\cdot(x_1,y_1)}{\text{Area of }\mathscr{R}_1 + \text{Area of }\mathscr{R}_2} = \frac{\frac{ab}{3}\cdot\big(\frac{3a}{4},\frac{3b}{10}\big) + \frac{2ab}{3}\big(\frac{3a}{8},\frac{3b}{5}\big)}{ab} = \Big(\frac{a}{2},\frac{b}{2}\Big)\,.$$ □