Calculus MA1002-B Quiz 09

National Central University, May. 19 2020

學號：____________________ 姓名：____________________

Problem 1. (3pts) Use the chain rule of functions of several variables to compute $\frac{dw}{dt}$, where $w = 2ye^x - \arcsin z$, $x = \ln(t^2+1)$, $y = \arctan t$, $z = \sin t$.

Solution. By the chain rule,

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= 2ye^x \cdot \frac{2t}{t^2+1} + 2e^x \cdot \frac{1}{1+t^2} - \frac{1}{\sqrt{1-z^2}} \cdot \cos t \\ &= 2\arctan t(t^2+1) \cdot \frac{2t}{t^2+1} + 2(t^2+1) \cdot \frac{1}{1+t^2} - \frac{1}{\cos^2 t}\cos t \\ &= 4t\arctan t + 2 - \frac{\cos t}{|\cos t|}\,.\end{aligned}$$

□

Problem 2. (4pts) Suppose that we substitute spherical coordinates $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ in a differentiable function $w = f(x,y,z)$. Show that

$$\frac{1}{\rho}\frac{\partial w}{\partial \theta} = -f_x\sin\theta\sin\phi + f_y\cos\theta\sin\phi \quad \text{and} \quad \frac{1}{\rho}\frac{\partial w}{\partial \phi} = f_x\cos\theta\cos\phi + f_y\sin\theta\cos\phi - f_z\sin\phi\,.$$

Proof. By the chain rule,

$$\begin{aligned}\frac{\partial w}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = f_x \cdot \big[\rho(-\sin\theta)\sin\phi\big) + f_y\rho\cos\theta\sin\phi + f_z \cdot 0 \\ &= \rho\big[-f_x\sin\theta\sin\phi + f_y\cos\theta\sin\theta\big]\,, \\ \frac{\partial w}{\partial \phi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi} = f_x\big(\rho\cos\theta\cos\phi\big) + f_y(\rho\sin\theta\cos\phi) + f_z\big[\rho(-\sin\phi)\big] \\ &= \rho\big[f_x\cos\theta\cos\phi + f_y\sin\theta\cos\phi - f_z\sin\phi\big]\,.\end{aligned}$$

The desired results are obtained by dividing both sides through by ρ. □

Problem 3. (3pts) Let $f(x,) = \sqrt[3]{xy}$. Show that f_x and f_y exist at the origin but that the directional derivatives at the origin in all other directions do not exist.

Proof. By the definition of partial derivatives,

$$\begin{aligned}f_x(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0\,, \\ f_y(0,0) &= \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0\,;\end{aligned}$$

thus f_x and f_y both exist at the origin. Let $\boldsymbol{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then

$$(D_{\boldsymbol{u}}f)(0,0) = \lim_{t\to 0}\frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} = \lim_{t\to 0^+}\frac{t^{\frac{2}{3}}\cos^{\frac{1}{3}}\theta\sin^{\frac{1}{3}}\theta}{t} = \lim_{t\to 0} t^{-\frac{1}{3}}\cos^{\frac{1}{3}}\theta\sin^{\frac{1}{3}}\theta$$

which does not exist unless $\theta = n\pi$ or $\theta = n\pi + \frac{\pi}{2}$. Therefore, the directional derivative at the origin does not exist in all other directions. □