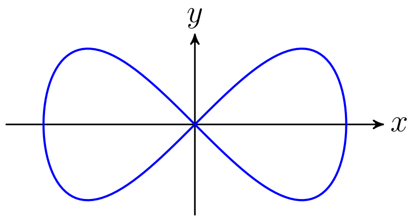
Calculus MA1002-B Quiz 06

National Central University, Apr. 28 2020

學號：____________________ 姓名：____________________

Problem 1. Let $\boldsymbol{r} : [0, 2\pi] \to \mathbb{R}^2$ be a vector-valued function defined by $\boldsymbol{r}(t) = \sin t\,\mathbf{i} + \sin t \cos t\,\mathbf{j}$. The image $\boldsymbol{r}([0, 2\pi])$ is a curve called figure eight.



1. (2pt) Determine the orientation of the parametrization; that is, find the direction of motion when t increases. Explain your answer and mark the direction on the figure above.
2. (3pts) Suppose that the polar representation of figure eight is given by $r^2 = f(\theta)$. Find f.
3. (2pts) Find all the horizontal tangent lines of figure eight.
4. (3pts) Find the area of the region enclosed by figure eight.

Solution. 1. First $\boldsymbol{r}(0) = \mathbf{0}$; thus the starting point of the curve (given by this parametrization) is the origin.

(a) When t increases in the interval $\left[0, \frac{\pi}{2}\right]$, the x-coordinate of $\boldsymbol{r}(t)$ increases and the y-coordinate of $\boldsymbol{r}(t)$ stays non-negative).

(b) When t increases further in the interval $\left[\frac{\pi}{2}, \pi\right]$, the x-coordinate of $\boldsymbol{r}(t)$ decreases and the y-coordinate of $\boldsymbol{r}(t)$ stays non-positive).

(c) When t increases further in the interval $\left[\frac{\pi}{2}, \pi\right]$, the x-coordinate of $\boldsymbol{r}(t)$ decreases and the y-coordinate of $\boldsymbol{r}(t)$ stays non-negative).

(d) When t increases further in the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, the x-coordinate of $\boldsymbol{r}(t)$ increases and the y-coordinate of $\boldsymbol{r}(t)$ stays non-positive).

2. Suppose (x, y) is on figure eight. Then $x = \sin t$ and $y = \sin t \cos t$ for some $t \in [0, 2\pi]$. Then $x^2 - y^2 = \sin^2 t - \sin^2 t \cos^2 t = \sin^2 t(1 - \cos^2 t) = \sin^4 t$ which implies that $y^2 = x^2 - x^4$. Therefore, if (x, y) is on figure eight, $y^2 = x^2(1-x^2)$. Therefore, if (r, θ) is the polar coordinate, then $r^2 \sin^2 \theta = r^2 \cos^2 \theta(1 - r^2 \cos^2 \theta)$; thus $r^2 = (1 - \tan^2 \theta) \sec^2 \theta = (2 - \sec^2 \theta) \sec^2 \theta$; thus $f(\theta) = (2 - \sec^2 \theta) \sec^2 \theta$.

3. At the tangent point, $y'(t) = \cos 2t = 0$ which implies that $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$. This gives two horizontal tangent lines: $y = \pm\frac{\sqrt{2}}{2}$.

4. The desired area is twice the area of the left part (whose parametrization gives counter-clockwise motion). Using the area formula, the area of the left part is given by
$$-\int_{\pi}^{2\pi} x'(t)y(t)\,dt = -\int_{\pi}^{2\pi} \cos t \sin t \cos t\,dt = \int_{\pi}^{2\pi} \cos^2 t\,d\cos t = \frac{1}{3}\cos^3 t\Big|_{t=\pi}^{t=2\pi} = \frac{2}{3}.$$
Therefore, the area of the region enclosed by figure eight is $\frac{4}{3}$. □