Calculus MA1002-B Quiz 04

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Problem 1. (3pts) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}$.

Solution. Since $\lim_{n\to\infty} \frac{1/(n2^n)}{1/\big[(n+1)2^{n+1}\big]} = 2$, we find that the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n2^n}$ is 2. Consider the convergence of the given power series at the end-points $-1-2=-3$ and $-1+2=1$.

1. Since a p-series converges if and only if $p>1$, we find that $\sum_{n=1}^{\infty} \frac{(1+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
2. On the other hand, since $\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series and $\frac{1}{n} \to 0$ as $n\to\infty$, we find that $\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n2^n}$ converges.

Therefore, the interval of convergence is $[-2,0)$. □

Problem 2. (3pts) Show that the power series $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ satisfies the differential equation

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0\,.$$

Proof. By the differentiation of power series,

$$y'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n}(n!)^2}\,, \quad y''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{2^{2n}(n!)^2}\,;$$

thus

$$\begin{aligned} x^2 y''(x) + xy'(x) + x^2 y(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{2^{2(n-1)}\big[(n-1)!\big]^2} \\ &= \sum_{n=1}^{\infty} \frac{2n(2n-1)+2n-4n^2}{2^{2n}(n!)^2} (-1)^n x^{2n} = 0\,. \end{aligned}$$

□

Problem 3. (4pts) Find the power series $x(t) = \sum_{n=0}^{\infty} a_n t^n$ that satisfies

$$x''(t) - x(t) = 0\,, \qquad x(0)=0, \quad x'(0)=1\,.$$

Solution. By the differentiation of power series,

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n\,, \quad x''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n\,;$$

thus if x satisfies the differential equation $x''(t) + 4x(t) = 0$, we must have

$$\sum_{n=0}^{\infty} \big[(n+2)(n+1) a_{n+2} - a_n\big] t^n = 0\,.$$

Therefore, $(n+2)(n+1)a_{n+2} = a_n$ for all $n \in \mathbb{N} \cup \{0\}$; thus $a_{n+2} = \dfrac{a_n}{(n+2)(n+1)}$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $x(0) = 0$ and $x'(0) = 1$ imply that $a_0 = 0$ and $a_1 = 1$. Therefore, $a_2 = a_4 = a_6 = \cdots = a_{2n} = \cdots = 0$ for all $n \in \mathbb{N}$, and

$$a_{2n+1} = \frac{a_{2n-1}}{(2n+1)(2n)} = \frac{a_{2n-3}}{(2n+1)(2n)(2n-1)(2n-2)} = \frac{1}{(2n+1)!} \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,.$$

As a consequence, $x(t) = \sum\limits_{n=0}^{\infty} \dfrac{t^{2n+1}}{(2n+1)!} = \dfrac{e^t - e^{-t}}{2}$. □