Calculus MA1002-B Quiz 03

National Central University, Mar. 31 2020

學號：____________________ 姓名：____________________

Problem 1. (3pts) State Taylor theorem (for functions of one variable).

Solution. Let $f:(a,b)\to\mathbb{R}$ be $(n+1)$-times differentiable, and $c\in(a,b)$. Then for each $x\in(a,b)$, there exists ξ between x and c such that

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+R_n(x)\,, \tag{0.1}$$

where Lagrange form of the remainder $R_n(x)$ is given by

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\,. \qquad \square$$

Problem 2. (4pts) Find the third Taylor polynomial for the function $f(x)=(\arctan x)^2$ about 0.

Solution. First we compute $f'(0)$, $f''(0)$ and $f'''(0)$. By the chain rule,

$$f'(x)=2(\arctan x)\frac{1}{1+x^2}=\frac{2\arctan x}{1+x^2}\,.$$

Then the quotient rule implies that

$$f''(x)=\frac{\frac{2}{1+x^2}(1+x^2)-4x\arctan x}{(1+x^2)^2}=\frac{2-4x\arctan x}{(1+x^2)^2}\,;$$

thus

$$f'''(x)=\frac{-4(\arctan x+\frac{x}{1+x^2})(1+x^2)^2-(2-4x\arctan x)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4}\,.$$

Therefore, $f(0)=0$, $f'(0)=0$, $f''(0)=2$ and $f^{(3)}(0)=0$; thus the third Taylor polynomial for f about 0 is

$$P_3(x)=0+0\cdot(x-0)+\frac{2}{2!}(x-0)^2+\frac{0}{3!}(x-0)^3=x^2\,. \qquad \square$$

Problem 3. (3pts) Let $f:(a,b)\to\mathbb{R}$ be a twice differentiable function such that $\big|f'(x)\big|\geqslant K$ and $\big|f''(x)\big|\leqslant M$ for all $x\in(a,b)$, where K,M are positive real numbers. Show that if $f(r)=0$ and $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$ for all $n\geqslant 1$, then

$$|x_{n+1}-r|\leqslant\frac{M}{2K}|x_n-r|^2 \qquad \forall\, n\geqslant 1\,.$$

Proof. By Taylor's theorem, $f(x)=f(x_n)+f'(x_n)(x-x_n)+\dfrac{f''(\xi)}{2}(x-x_n)$ for some ξ between x and x_n. Then

$$0=f(x_n)+f(x_n)(r-x_n)+\frac{f''(\xi)}{2}(r-x_n)^2\,;$$

thus if $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$,

$$|x_{n+1}-r|=\Big|x_n-r-\frac{f(x_n)}{f'(x_n)}\Big|=\Big|\frac{f'(x_n)(x_n-r)-f(x_n)}{f'(x_n)}\Big|=\Big|\frac{f''(\xi)(r-x_n)^2}{2f'(x_n)}\Big|\leqslant\frac{M}{2K}|x_n-r|^2\,. \qquad \square$$