Calculus MA1002-B Quiz 02

National Central University, Mar. 24 2020

學號：____________________ 姓名：____________________

Problem 1. (2pts) Determine whether the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent or divergent.

Solution. Let $a_n = \frac{n!}{n^n}$. Then $a_n > 0$ and the fact that $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$ implies that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = e^{-1} < 1\,.$$

By the ratio test, we conclude that $\sum_{n=2}^{\infty} a_n$ converges absolutely; thus $\sum_{n=2}^{\infty} a_n$ is convergent. □

Problem 2. (3pts) Determine whether the series $\sum_{n=2}^{\infty} \left(1-\frac{1}{n}-\frac{1}{n^2}\right)^{n^2}$ is convergent or divergent.

Solution. Let $a_n = \left(1-\frac{1}{n}-\frac{1}{n^2}\right)^{n^2}$. Then $a_n > 0$ for all $n \geqslant 2$ and $\sqrt[n]{a_n} = \left(1-\frac{1}{n}-\frac{1}{n^2}\right)^n$. Moreover,

$$\left(1-\frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}+1} \leqslant \sqrt[n]{a_n} \leqslant \left(1-\frac{1}{n}\right)^n \qquad \forall\, n \geqslant 2\,.$$

By the fact that $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = e^{-1}$, we obtain from the Squeeze Theorem that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = e^{-1} < 1\,.$$

By the root test, we conclude that $\sum_{n=2}^{\infty} a_n$ converges absolutely; thus $\sum_{n=2}^{\infty} a_n$ is convergent. □

Problem 3. (2pts) Determine whether the series $\sum_{n=1}^{\infty} \sin^n \frac{1}{\sqrt{n}}$ is convergent or divergent.

Solution. Let $a_n = \sin^n \frac{1}{\sqrt{n}}$ and $b_n = \frac{1}{\sqrt{n}^n}$. By the fact that $0 \leqslant \sin x \leqslant x$ for all $x \geqslant 0$, we have $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$. Moreover, $b_n \leqslant 2^{-n}$ for all $n \geqslant 4$. Since $\sum_{n=4}^{\infty} 2^{-n}$ is a geometric series with common ratio $\frac{1}{2}$, we find that $\sum_{n=4}^{\infty} 2^{-n}$ converges. By the direct comparison test, $\sum_{n=1}^{\infty} b_n$ converges; thus $\sum_{n=1}^{\infty} a_n$ is convergent. □

Problem 4. (3pts) Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$ is convergent or divergent.

Solution. Let $a_n = \frac{\sin^2 n}{n}$ and $b_n = \frac{\cos(2n)}{n}$. Then $a_n = \frac{1}{2n} - b_n$. Since

$$\Big|\sum_{k=1}^{n} \cos(2k)\Big| = \Big|\sum_{k=1}^{n} \frac{\sin(2k+1)-\sin(2k-1)}{2\sin 1}\Big| = \Big|\frac{\sin(2n+1)-\sin 1}{2\sin 1}\Big| \leqslant \frac{1}{|\sin 1|}\,,$$

the Dirichlet test implies that $\sum_{n=1}^{\infty} b_n$ is convergent. Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent, we find that $\sum_{n=1}^{\infty} a_n$ is divergent. □