Calculus MA1002-B Quiz 01

National Central University, Mar. 10 2020

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Problem 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

1. (2pts) Write down the definition of the convergence of $\{a_n\}_{n=1}^{\infty}$.
2. (2pts) Write down the definition of the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$.

Solution. 1. The sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $N > 0$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n \geqslant N .$$

2. The series $\sum\limits_{n=1}^{\infty} a_n$ is said to converge if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$, where $S_n = \sum\limits_{k=1}^{n} a_k$, converges. □

Problem 2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers defined recursively by $a_{n+1} = \dfrac{4}{3+a_n}$ with $a_1 = 0$. Show that $\{a_n\}_{n=1}^{\infty}$ converges to 1 by completing the following:

1. (2pts) Show that $a_{n+1} - 1 = \dfrac{1-a_n}{3+a_n}$ for all $n \in \mathbb{N}$ and conclude that $|a_{n+1} - 1| \leqslant \dfrac{1}{3}|a_n - 1|$ for all $n \in \mathbb{N}$.
2. (2pts) Show that $|a_n - 1| \leqslant \left(\dfrac{1}{3}\right)^{n-1}|a_1 - 1|$ and conclude that $\lim\limits_{n\to\infty} a_n = 1$.

Proof. First we observe $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Since

$$a_{n+1} - 1 = \frac{4}{3+a_n} - 1 = \frac{1-a_n}{3+a_n} ,$$

we find that $\big|a_{n+1} - 1\big| = \dfrac{1}{3+a_n}|a_n - 1| \leqslant \dfrac{1}{3}|a_n - 1|$; thus

$$\big|a_n - 1\big| \leqslant \frac{1}{3}|a_{n-1} - 1| \leqslant \frac{1}{3}\cdot\frac{1}{3}|a_{n-2} - L| \leqslant \cdots \leqslant \left(\frac{1}{3}\right)^{n-1}|a_1 - 1| .$$

Since $\dfrac{1}{3} < 1$, by the squeeze theorem we find that $\lim\limits_{n\to\infty} |a_n - 1| = 0$. Therefore, $\lim\limits_{n\to\infty} a_n = 1$. □

Problem 3. (2pts) Determine whether the series $\sum\limits_{n=1}^{\infty} \dfrac{40n}{(2n-1)^2(2n+1)^2}$ converges or not.

Proof. Let $a_k = \dfrac{40k}{(2k-1)^2(2k+1)^2}$ and $b_k = \dfrac{5}{2k-1}$. Then $a_k = b_k - b_{k+1}$; thus the partial sum

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_k - b_{k+1}) = b_1 - b_{n+1} .$$

Since $\lim\limits_{n\to\infty} b_{n+1} = 0$, we find that $\lim\limits_{n\to\infty} S_n = b_1$; thus the series $\sum\limits_{n=1}^{\infty} \dfrac{40n}{(2n-1)^2(2n+1)^2}$ converges. □