Exercise Problem Sets 13

Jun. 12. 2020

Problem 1. Evaluate the following iterated integrals.

(1) $\int_{-1}^{1}\Big(\int_{0}^{1} y\exp(x^2+y^2)\,dx\Big)dy$ (2) $\int_{0}^{2}\Big(\int_{y}^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\Big)dy$ (3) $\int_{0}^{1}\Big(\int_{\sqrt{y}}^{1} \exp(x^3)\,dx\Big)dy$

(4) $\int_{0}^{1}\Big(\int_{y}^{1} \frac{1}{1+x^4}\,dx\Big)dy$ (5) $\int_{0}^{4}\Big(\int_{\frac{x}{2}}^{2} \sin(y^2)\,dy\Big)dx$ (6) $\int_{0}^{4}\Big(\int_{\sqrt{x}}^{2} \frac{1}{y^3+1}\,dy\Big)\,dx$

(7) $\int_{0}^{2}\Big(\int_{x}^{2} x\sqrt{1+y^3}\,dy\Big)dx$ (8) $\int_{0}^{2}\Big(\int_{\frac{y}{2}}^{1} \exp(x^2)\,dx\Big)dy$ (9) $\int_{0}^{1}\Big(\int_{0}^{1} \frac{y}{1+x^2y^2}\,dx\Big)\,dy$

(10) $\int_{0}^{\frac{\pi}{2}}\Big(\int_{x}^{\frac{\pi}{2}} \frac{\sin y}{y}\,dy\Big)dx$ (11) $\int_{0}^{2}\Big(\int_{y^2}^{4} \sqrt{x}\sin x\,dx\Big)dy$ (12) $\int_{0}^{2}\Big(\int_{0}^{4-x^2} \frac{xe^{2y}}{4-y}dy\Big)dx$

(13) $\int_{0}^{1}\Big(\int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\,dx\Big)dy$ (14) $\int_{-5}^{5}\Big[\int_{0}^{\sqrt{25-x^2}}\Big(\int_{0}^{\frac{1}{x^2+y^2}} \sqrt{x^2+y^2}\,dz\Big)dy\Big]dx$

(15) $\int_{0}^{4}\Big[\int_{0}^{1}\Big(\int_{2y}^{2} \frac{2\cos(x^2)}{\sqrt{z}}\,dx\Big)dy\Big]dz$ (16) $\int_{0}^{1}\Big[\int_{0}^{1}\Big(\int_{x^2}^{1} xz\exp(zy^2)\,dy\Big)dx\Big]dz$

(17) $\int_{0}^{1}\Big[\int_{\sqrt[3]{z}}^{1}\Big(\int_{0}^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,dx\Big)dy\Big]dz$ (18) $\int_{0}^{2}\Big[\int_{0}^{4-x^2}\Big(\int_{0}^{x} \frac{\sin(2z)}{4-z}\,dy\Big)dz\Big]dx$

Problem 2. Evaluate the double integral $\iint\limits_R f(x,y)\,dA$ with the following f and R.

(1) $f(x,y)=y^2e^{xy}$, and R is the region bounded by $y=x$, $y=4$ and $x=0$.

(2) $f(x,y)=xy$, and R is the region bounded by the line $y=x-1$ and parabola $y^2=2x+6$.

(3) $f(x,y)=\sin^4(x+y)$, and R is the triangle enclosed by the lines $y=0$, $y=2x$, and $x=1$.

(4) $f(x,y)=x^2+x^2y^3-y^2\sin x$, and $R=\big\{(x,y)\,\big|\,|x|+|y|\leqslant 1\big\}$.

(5) $f(x,y)=|x|+|y|$, and $R=\big\{(x,y)\,\big|\,|x|+|y|\leqslant 1\big\}$.

(6) $f(x,y)=xy$, and R is the region in the first quadrant bounded by curves $x^2+y^2=4$, $x^2+y^2=9$, $x^2-y^2=1$ and $x^2-y^2=4$.

(7) $f(x,y)=x$, and R is the region in the first quadrant bounded by curves $4x^2-y^2=4$, $4x^2-y^2=16$, $y=x$ and the x-axis.

(8) $f(x,y)=\exp(-x^2-4y^2)$, and $R=\big\{(x,y)\,\big|\,x^2+4y^2\leqslant 1\big\}$.

(9) $f(x,y)=\exp\big(\frac{2y-x}{2x+y}\big)$, and R is the trapezoid with vertices $(0,2)$, $(1,0)$, $(4,0)$ and $(0,8)$.

Problem 3. Evaluate the triple integral $\iiint\limits_D f(x,y,z)\,dV$ with the following f and D.

(1) $f(x,y,z) = x - y + z^2$, and D is the solid region bounded above by $z = 1 + x^2 + y^2$, bounded below by $z = 0$, and inside $x^2 + y^2 = 4$.

(2) $f(x,y,z) = 1$, and D is the solid region bounded by $z = x^2 + y^2$, $x^2 + y^2 = 4$ and $z = 0$.

(3) $f(x,y,z) = 1$, and $D = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\right\}$, where $a, b, c > 0$.

Problem 4. Evaluate the integral $\int_0^2 \big[\arctan(\pi x) - \arctan x\big]\,dx$ by converting the integral into a double integral and evaluating the double integral by changing the order of integration.

Problem 5. Let a, b be positive constants. Evaluate the integral $\int_0^a \Big(\int_0^b \exp\big(\max\{b^2x^2, a^2y^2\}\big)\,dy\Big)\,dx$.

Problem 6. Show that if $\lambda > \frac{1}{2}$, there does not exist a real-valued continuous function u such that for all x in the closed interval $[0,1]$,

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y-x)\,dy\,.$$

Hint: Assume the contrary that there exists such a function u. Integrate the equation above on the interval $[0,1]$.

Problem 7. Find the surface area for the portion of the surface $z = xy$ that is inside the cylinder $x^2 + y^2 = 1$.

Problem 8. Let Σ be a parametric surface parameterized by

$$\boldsymbol{r}(u,v) = X(u,v)\mathbf{i} + Y(u,v)\,\mathbf{j} + Z(u,v)\mathbf{k}\,, \quad (u,v) \in R\,.$$

Define $E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u$, $F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v$ and $G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v$. Show that

$$\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|^2 = EG - F^2\,.$$

Hint: You can try to make use of ε_{ijk}, the permutation symbol.
Remark: This quantity $EG - F^2$ is called the first fundamental form (associated with the parametrization $\boldsymbol{r}$).

Problem 9. Let $k > 0$ be a constant. Show that the surface area of the cone $z = k\sqrt{x^2 + y^2}$ that lies above the circular region $x^2 + y^2 \leqslant r^2$ in the xy-plane is $\pi r^2 \sqrt{k^2 + 1}$ by the following methods:

1. Use the formula $\iint\limits_R \sqrt{1 + \|(\nabla f)(x,y)\|^2}\,dA$ directly.

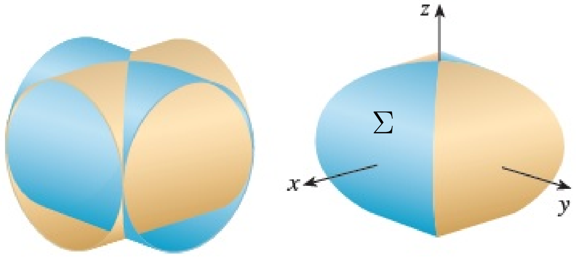
2. Find a parametrization of the cone above using r, θ (from the polar coordinate) as the parameters and make use of the formula $\iint\limits_D \big\|(\boldsymbol{r}_r \times \boldsymbol{r}_\theta)(r,\theta)\big\|\,d(r,\theta)$.

Problem 10. Let Σ be the surface formed by rotating the curve

$$C = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x = \cos z, y = 0, -\frac{\pi}{2} \leqslant z \leqslant \frac{\pi}{2} \right\}$$

about the z-axis. Find a parametrization for Σ and compute its surface area.

Problem 11. The figure below shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Let Σ be the part shown in the figure.



(1) Find the area of Σ using the formula $\displaystyle\iint_R \sqrt{1 + \|(\nabla f)(x, y)\|^2}\, dA$.

(2) Parameterize Σ using θ, z as parameters (from the cylindrical coordinate) and find the area of this surface using the formula $\displaystyle\iint_D \|(\boldsymbol{r}_\theta \times \boldsymbol{r}_z)(\theta, z)\|\, d(\theta, z)$.

(3) Parameterize Σ using θ, ϕ as parameters (from the spherical coordinate) and find the area of this surface using the formula $\displaystyle\iint_D \|(\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi)(\theta, \phi)\|\, d(\theta, \phi)$.

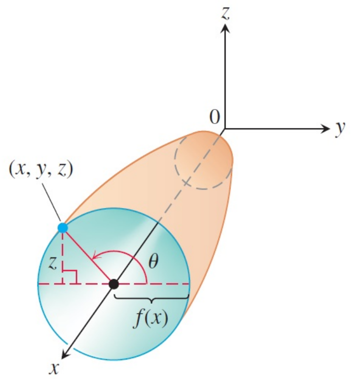
(3) Find the volume of this intersection using triple integrals.

Problem 12. Let Σ be the surface obtained by rotating the smooth curve $y = f(x)$, $a \leqslant x \leqslant b$ about the x-axis, where $f(x) > 0$.

1. Show that

$$\boldsymbol{r}(x, \theta) = x\mathbf{i} + f(x)\cos\theta\,\mathbf{j} + f(x)\sin\theta\,\mathbf{k}, \quad (x, \theta) \in [a, b] \times [0, 2\pi],$$

is a parametrization of Σ, where θ is the angle of rotation about the x-axis (see the accompanying figure).

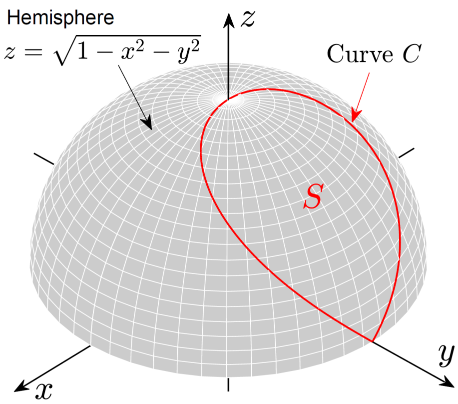


2. Show that the surface area of Σ is

$$\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

using the formula $\displaystyle\iint_D \|(\boldsymbol{r}_x \times \boldsymbol{r}_\theta)(x,\theta)\|\,d(x,\theta)$.

Problem 13. Let S be the subset of the upper hemisphere $z = \sqrt{1-x^2-y^2}$ enclosed by the curve C shown in the figure below



where each point of C corresponds to some point $(\cos t\sin t, \sin^2 t, \cos t)$ with $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Find the surface of S via the following steps:

(1) Let R be the region obtained by projecting S onto the xy-plane along the z-axis. Suppose that R can be expressed as $R = \left\{(x,y)\,\middle|\, c \leqslant y \leqslant d, g_1(y) \leqslant x \leqslant g_2(y)\right\}$. Find c, d and g_1, g_2, and find the surface area of S using the formula $\displaystyle\iint_R \sqrt{1+\|(\nabla f)(x,y)\|^2}\,dA$.

(2) The surface S is a parametric surface parameterized by

$$S = \left\{\boldsymbol{r}\,\middle|\, \boldsymbol{r} = \cos\theta\sin\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\phi\,\mathbf{k} \text{ for some } (\theta,\phi) \in D\right\}.$$

Find the domain D inside the rectangle $[0,2\pi]\times[0,\pi]$, and find the surface area of S using the formula $\displaystyle\iint_D \|(\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi)(\theta,\phi)\|\,d(\theta,\phi)$.

Problem 14. Rewrite the following iterated integrals as an equivalent iterated integral in the five other orders.

(1) $\displaystyle\int_0^1\left[\int_y^1\left(\int_0^y f(x,y,z)\,dz\right)dx\right]dy$

(2) $\displaystyle\int_0^1\left[\int_y^1\left(\int_0^z f(x,y,z)\,dx\right)dz\right]dy$

(3) $\displaystyle\int_0^1\left[\int_0^{1-x^2}\left(\int_0^{1-x} f(x,y,z)\,dy\right)dz\right]dx$

(4) $\displaystyle\int_0^3\left[\int_0^x\left(\int_0^{9-x^2} f(x,y,z)\,dz\right)dy\right]dx$

(5) $\displaystyle\int_0^1\left[\int_{\sqrt{x}}^1\left(\int_0^{1-y} f(x,y,z)\,dz\right)dy\right]dx$

(6) $\displaystyle\int_{-1}^1\left[\int_{x^2}^1\left(\int_0^{1-y} f(x,y,z)\,dz\right)dy\right]dx$

Problem 15. Find volume of the solid that lies under $z = x^2 + y^2$ and above the region R in the xy-plane bounded by the line $y = 2x$ and parabola $y = x^2$.

Problem 16. Evaluate the triple integral $\iiint\limits_D dV$, where D is bounded by $z = x^2 + y^2$, $x^2 + y^2 = 4$ and $z = 0$.

Problem 17. Evaluate the double integral $\iint\limits_R \arctan \frac{y}{x}\, dA$ using the polar coordinate, where

$$R = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 1 \leqslant x^2 + y^2 \leqslant 4, 0 \leqslant y \leqslant x\right\}.$$

Problem 18. Evaluate the triple integral $\iiint\limits_D x \exp(x^2 + y^2 + z^2)\, dV$, where D is the portion of the unit ball $x^2 + y^2 + z^2 \leqslant 1$ that lies in the first octant.

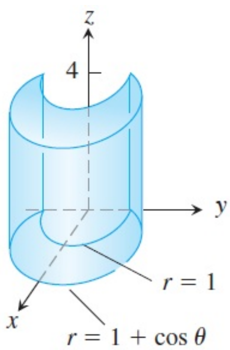
Problem 19. Evaluate the triple integral $\iiint\limits_D \sqrt{x^2 + y^2 + z^2}\, dV$, where D is the region lying above the cone $z = \sqrt{x^2 + y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

Problem 20. Use the cylinder coordinate to find the volume of the ball $x^2 + y^2 + z^2 = a^2$.

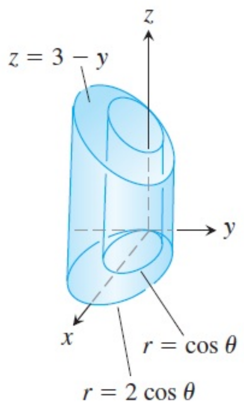
Problem 21. Use the spherical coordinate to find the volume of the cylindricality $x^2 + y^2 = r^2$, where $0 \leqslant z \leqslant h$.

Problem 22. Compute the volume of D given below using triple integrals in cylindrical coordinates.

(1) D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.

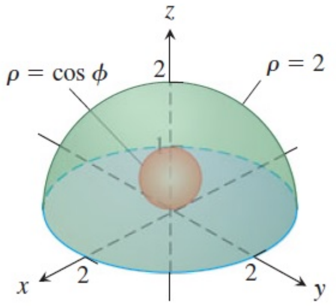


(2) D is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$.

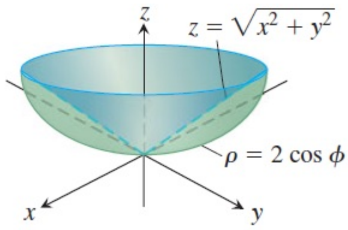


Problem 23. Compute the volume of D given below using triple integrals in spherical coordinates.

(1) D is the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geqslant 0$.



(2) D is the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2+y^2}$.



Problem 24. Convert the integral

$$\int_{-1}^{1}\Big[\int_{0}^{\sqrt{1-y^2}}\Big(\int_{0}^{x}(x^2+y^2)\,dz\Big)dx\Big]dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

Problem 25. Find the integrals given below with specific change of variables.

(1) Find $\displaystyle\int_0^2\Big(\int_{\frac{y}{2}}^{\frac{y+4}{2}} y^3(2x-y)e^{(2x-y)^2}\,dx\Big)dy$ using change of variables $x = u + \dfrac{1}{2}v$, $y = v$.

(2) Find $\displaystyle\int_1^2\Big(\int_{\frac{1}{y}}^{y}(x^2+y^2)\,dx\Big)dy + \int_2^4\Big(\int_{\frac{y}{4}}^{\frac{4}{y}}(x^2+y^2)\,dx\Big)dy$ using change of variables $x = \dfrac{u}{v}$, $y = uv$.

(3) Find $\displaystyle\int_0^1\Big(\int_0^{2\sqrt{1-x}}\sqrt{x^2+y^2}\,dy\Big)dx$ using change of variables $x = u^2 - v^2$, $y = 2uv$.

(4) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Find $\displaystyle\iint_R\Big(\sqrt{\frac{y}{x}} + \sqrt{xy}\Big)\,dA$ using the change of variables $x = \dfrac{u}{v}$, $y = uv$.

(5) Let D be the solid region in xyz-space defined by

$$D = \big\{(x,y,z)\,\big|\,1 \leqslant x \leqslant 2, 0 \leqslant xy \leqslant 2, 0 \leqslant z \leqslant 1\big\}.$$

Find $\displaystyle\iiint_D (x^2y + 3xyz)\,dV$ using change of variables $u = x$, $v = xy$, $w = 3z$.

Problem 26. Evaluate the double integral $\iint\limits_R (x+y)e^{x^2-y^2}\,dA$, where R is rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

Problem 27. Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Show that

$$\iint\limits_R f(x+y)\,dA = \int_0^1 uf(u)\,du\,.$$

Problem 28. Let A be the area of the region in the first quadrant bounded by the line $y=\frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2+y^2=1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y=mx$, the y-axis, and the ellipse $\frac{1}{9}x^2+y^2=1$.
Hint: Try to make change of variables so that the computation of the area of the region in the first quadrant bounded by the line $y=mx$, the y-axis, and the ellipse $\frac{1}{9}x^2+y^2=1$ looks the same as the former one.