Exercise Problem Sets 10

May. 15. 2020

Problem 1. Use the chain rule for functions of several variables to compute $\frac{dz}{dt}$ or $\frac{dw}{dt}$.

(1) $z = \sqrt{1+xy}$, $x = \tan t$, $y = \arctan t$.

(2) $w = x \exp\left(\frac{y}{z}\right)$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$.

(3) $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$.

(4) $w = xy\cos z$, $x = t$, $y = t^2$, $z = \arccos t$.

(5) $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \arctan t$, $z = e^t$.

Problem 2. Use the chain rule for functions of several variables to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(1) $z = \arctan(x^2+y^2)$, $x = s\ln t$, $y = te^s$.

(2) $z = \arctan\frac{x}{y}$, $x = s\cos t$, $y = s\sin t$.

(3) $z = e^x \cos y$, $x = st$, $y = s^2 + t^2$.

Problem 3. Assume that $z = f\left(ts^2, \frac{s}{t}\right)$, $\frac{\partial f}{\partial x}(x,y) = xy$, $\frac{\partial f}{\partial y}(x,y) = \frac{x^2}{2}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Problem 4. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at given points.

(1) $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$, $(x,y,z) = (\pi,\pi,\pi)$.

(2) $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$, $(x,y,z) = (1, \ln 2, \ln 3)$.

(3) $z = e^x\cos(y+z)$, $(x,y,z) = (0,-1,1)$.

Problem 5. Let f be differentiable, and $z = \frac{1}{y}\big[f(ax+y) + g(ax-y)\big]$. Show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{y^2}\frac{\partial}{\partial y}\left(y^2\frac{\partial z}{\partial y}\right).$$

Problem 6. Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $z = f(x,y)$.

(1) Show that $\frac{\partial z}{\partial r} = f_x\cos\theta + f_y\sin\theta$ and $\frac{1}{r}\frac{\partial r}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta$.

(2) Solve the equations in part (1) to express f_x and f_y in terms of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

(3) Show that $(f_x)^2 + (f_y)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

(4) Suppose in addition that f_x and f_y are differentiable. Show that

$$f_{xx} + f_{yy} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.$$

Problem 7. （此題不在本週講授的課程範圍，算是增加知識用，也為了下一題做準備）Let R be an open region in $\mathbb{R}^2$ and $f : R \to \mathbb{R}$ be a real-valued function. In class we have talked about the differentiability of f. For $k \geqslant 2$, the k-times differentiability of f is defined inductively: for $k \in \mathbb{N}$, f is said to be $(k+1)$-times differentiable at (a,b) if the k-th partial derivative $\dfrac{\partial^k f}{\partial x^{k-j}\partial y^j}$ is differentiable at (a,b) for all $0 \leqslant j \leqslant k$ $\big($note that in order to achieve this, $\dfrac{\partial^k f}{\partial x^{k-j}\partial y^j}$ has to be defined in a neighborhood of (a,b) for all $0 \leqslant j \leqslant k\big)$. f is said to be k-times differentiable on R if f is k-times differentiable at (a,b) for all $(a,b) \in R$. f is said to be k-times continuously differentiable on R if the k-th partial derivative $\dfrac{\partial^k f}{\partial x^{k-j}\partial y^j}$ is continuous at (a,b) for all $0 \leqslant j \leqslant k$.

(1) Show that if f is $(k+1)$-times differentiable on R, then f is k-times continuously differentiable on R.

(2) Show that if f is k-times continuously differentiable on R, then f is k-times differentiable on R.

Hint: In this problem the following fact is used (without proving yet):

if f_x and f_y are continuous at (a,b), then f is differentiable at (a,b).

Remark: In the course "Introduction to Mathematical Analysis (分析導論)", the differentiability of a function will be discussed more systematically.

Problem 8. In this problem we investigate the Taylor Theorem for functions of two variables. Let R be an open region in $\mathbb{R}^2$, $(a,b) \in R$, and $f : R \to \mathbb{R}$ be a $(n+1)$-times differentiable function for some $n \in \mathbb{N} \cup \{0\}$ (the $(n+1)$-times differentiability of functions of two variables is defined in Problem 7. Complete the following.

(1) For a given point $(x,y) \in R$, suppose that the line segment connecting (a,b) and (x,y) belongs to R; that is,

$$\Big\{(a + t(x-a))\mathbf{i} + (b + t(y-b))\mathbf{j} \,\Big|\, t \in [0,1]\Big\} \subseteq R.$$

Define $\boldsymbol{r}(t) = (a + t(x-a))\mathbf{i} + (b + t(y-b))\mathbf{j}$. Show that the function $g(t) = f(\boldsymbol{r}(t)) = f\big(a + t(x-a), b + t(y-b)\big)$ is $(n+1)$-times differentiable on I.

(2) Show that for $1 \leqslant k \leqslant n$,

$$g^{(k)}(t) = \sum_{j=0}^{k} C^k_j \frac{\partial^k f}{\partial x^{k-j}\partial y^j}(\boldsymbol{r}(t))(x-a)^{k-j}(y-b)^j.$$

(3) Show that if f is $(n+1)$-times continuously differentiable on R, then

$$f(x,y) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} C_j^k \frac{\partial^k f}{\partial x^{k-j} \partial y^j}(a,b)(x-a)^{k-j}(y-b)^j$$
$$+ \frac{1}{(n+1)!} \sum_{j=0}^{n+1} C_j^{n+1} \frac{\partial^{n+1} f}{\partial x^{n-j+1} \partial y^j}(c,d)(x-a)^{k-j}(y-b)^j$$

for some point (c,d) on the line segment connecting (a,b) and (x,y).

Hint: By the Taylor Theorem for functions of one variable, there exists $0 < \xi < 1$ such that

$$g(1) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} + g^{(n+1)}(\xi)\,.$$

(4) The polynomial $\sum\limits_{k=0}^{n} \frac{1}{k!} \sum\limits_{j=0}^{k} C_j^k \frac{\partial^k f}{\partial x^{k-j} \partial y^j}(a,b)(x-a)^{k-j}(y-b)^j$ is called the n-th Taylor polynomial for f at (a,b). Find the fourth Taylor polynomial for the following functions at $(0,0)$.

(a) $\sin(x^3 + y^4)$ (b) $\exp\big(x^2 + y^2\big)$ (c) $\ln\big(\cos(x^2 + y)\big)$

Problem 9. Let $f(x,) = \sqrt[3]{xy}$.

(1) Show that f is continuous at $(0,0)$.

(2) Show that f_x and f_y exist at the origin but that the directional derivatives at the origin in all other directions do not exist.

Problem 10. Let

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

(1) Show that the directional derivative of f at the origin exists in all directions $\boldsymbol{u}$, and

$$(D_{\boldsymbol{u}} f)(0,0) = \Big(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\Big) \cdot \boldsymbol{u}\,.$$

(2) Determine whether f is differentiable at $(0,0)$ or not.

Problem 11. Let $\boldsymbol{u} = (a,b)$ be a unit vector and f be twice continuously differentiable. Show that

$$D_{\boldsymbol{u}}^2 f = f_{xx} a^2 + 2 f_{xy} ab + f_{yy} b^2\,,$$

where $D_{\boldsymbol{u}}^2 f = D_{\boldsymbol{u}}(D_{\boldsymbol{u}} f)$.

Problem 12. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and that a, b are constants.

(1) $\nabla(au + bv) = a\nabla u + b\nabla v$.

(2) $\nabla(uv) = u\nabla v + v\nabla u$.

(3) $\nabla\left(\frac{u}{v}\right) = \dfrac{v\nabla u - u\nabla v}{v^2}$.

(4) $\nabla(u^n) = nu^{n-1}\nabla u$.

Problem 13. Show that the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1\,.$$

Problem 14. Show that the equation of the tangent plane to the elliptic paraboloid $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ at the point (x_0, y_0, z_0) can be written as

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}\,.$$