Exercise Problem Sets 7

Apr. 25. 2020

Problem 1. Let C be a curve parameterized by the vector-valued function $\boldsymbol{r}:[0,1]\to\mathbb{R}^2$,

$$\boldsymbol{r}(t)=\Big(\frac{e^t-e^{-t}}{e^t+e^{-t}},\frac{2}{e^t+e^{-t}}\Big),\quad 0\leqslant t\leqslant 1\,.$$

(1) Show that C is part of the unit circle centered at the origin.

(3) Find the length of the curve C.

Problem 2. Let C be the curve given by the parametric equations

$$x(t)=\frac{3+t^2}{1+t^2}\,,\qquad y(t)=\frac{2t}{1+t^2}$$

on the interval $t\in[0,1]$.

(1) In fact C is the graph of a function $y=f(x)$. Find f.

(2) Find the arc length of the curve C.

(3) Find the area of the surface formed by revolving the curve C about the y-axis.

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Let C be a simple closed curve in the plane parameterized by $\boldsymbol{r}:[a,b]\to\mathbb{R}^2$. Suppose that

1. $\boldsymbol{r}(t)=\big(x(t),y(t)\big)$ moves **counter-clockwise** (that is, the region enclosed by C is on the left-hand side when moving along C) as t increases.

2. There exists $c\in(a,b)$ such that x is strictly **increasing** on $[a,c]$ and is strictly **decreasing** on $[c,b]$ (this implies that every vertical line intersects with the curve C at at most two points)

3. $x'y$ is Riemann integrable on $[a,b]$ (for example, x is continuously differentiable on $[a,b]$).

In class we "prove" that under the assumptions above

$$\text{the area of the region enclosed by } C \text{ is } -\int_a^b x'(t)y(t)\,dt\,. \tag{0.1}$$

Similar argument can be applied to obtain that

$$\text{the area of the region enclosed by } C \text{ is } \int_a^b x(t)y'(t)\,dt\,. \tag{0.2}$$

if xy' is Riemann integrable on $[a, b]$ and every horizontal line intersects with the curve C at at most two points. Combining (0.1) and (0.2), we obtain that

$$\text{the area of the region enclosed by } C \text{ is } \frac{1}{2}\int_a^b \Big[x(t)y'(t) - x'(t)y(t)\Big]\,dt \tag{0.3}$$

provided that $x'y$ and xy' are Riemann integrable on $[a, b]$ and every vertical line and horizontal line intersects with the curve C at at most two points.

In general, **the restriction that every vertical line or horizontal line intersects with curve C at at most two points can be removed** from the condition for the use of (0.1), (0.2) and (0.3). We will treat this as a fact for we have proved a special case.

Using the convention that $\mathbf{u} \times \mathbf{v} = u_1v_2 - u_2v_1$ when $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ are vectors in the plane, (0.3) can be rewritten as

$$\text{the area of the region enclosed by } C \text{ is } \frac{1}{2}\int_a^b \boldsymbol{r}(t) \times \boldsymbol{r}'(t)\,dt\,. \tag{0.3'}$$

Without confusion, the area can also be computed by $\frac{1}{2}\int_a^b \boldsymbol{r}(t) \times d\boldsymbol{r}(t)$.

Example 0.1. Let C be the curve parameterized by $\boldsymbol{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then the area of the region enclosed by C can be computed by the following three ways:

1. Using (0.1),

$$-\int_0^{2\pi} \frac{d\cos t}{dt}\sin t\,dt = \int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt = \frac{1}{2}\Big(t - \frac{\sin(2t)}{2}\Big)\Big|_{t=0}^{t=2\pi} = \pi\,.$$

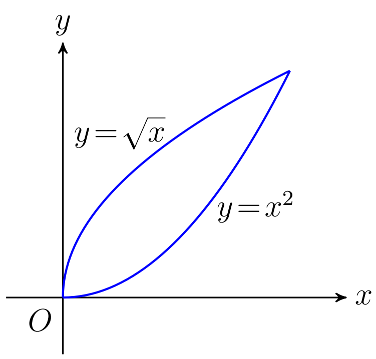
2. Using (0.2),

$$\int_0^{2\pi} \cos t\,\frac{d\sin t}{dt}\,dt = \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt = \frac{1}{2}\Big(t + \frac{\sin(2t)}{2}\Big)\Big|_{t=0}^{t=2\pi} = \pi\,.$$

3. Using (0.3),

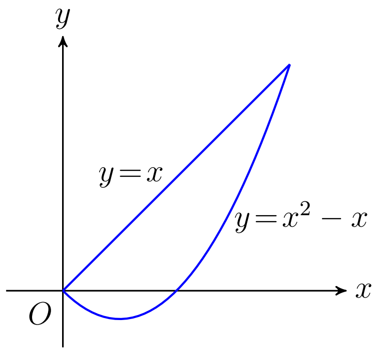
$$\frac{1}{2}\int_0^{2\pi} \Big(\cos t\frac{d\sin t}{dt} - \frac{d\cos t}{dt}\sin t\Big)\,dt = \frac{1}{2}\int_0^{2\pi} \big(\cos^2 t + \sin^2 t\big)\,dt = \frac{1}{2}\int_0^{2\pi} 1dt = \pi\,.$$

Problem 3. Give a parametrization of the simple closed curve C shown in the figure below



and find the area of the region enclosed by C using (0.1), (0.2) or (0.3).

Problem 4. Give a parametrization of the simple closed curve C shown in the figure below

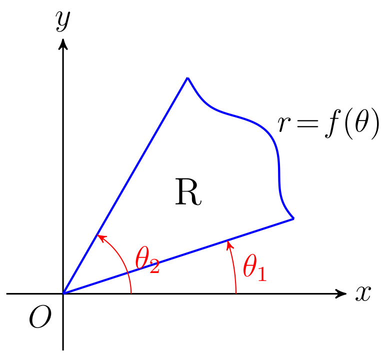


and find the area of the region enclosed by C using (0.1), (0.2) or (0.3).

Problem 5. Let $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$, and $f : [\theta_1, \theta_2] \to \mathbb{R}$ be a non-negative function. Consider the polar region

$$\mathrm{R} = \left\{(r, \theta) \,\middle|\, 0 \leqslant r \leqslant f(\theta), \theta \in [\theta_1, \theta_2]\right\}$$

shown in the figure below.



1. Give a parametrization of the boundary of the polar region R.

2. Using (0.3) to show that the area of the polar region R is given by

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} f(t)^2\, dt$$

Problem 6. Let C_1 be the polar graph of the polar function $r = 1 + \cos\theta$ (which is a cardioid), and C_2 be the polar graph of the polar function $r = 3\cos\theta$ (which is a circle). See the following figure for reference.

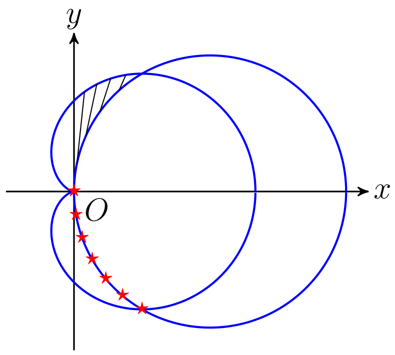


Figure 1: The polar graphs of the polar equations $r = 1 + \cos\theta$ and $r = 3\cos\theta$

(1) Find the intersection points of C_1 and C_2.

(2) Find the line L passing through the lowest intersection point and tangent to the curve C_2.

(3) Identify the curve marked by $\star$ on the θr-plane for $0 \leqslant \theta \leqslant 2\pi$.

(4) Find the area of the shaded region.

以下問題部份小題要用到第七章的觀念，為了題目的完整性一併呈現，但是與第十二章有關的是 (1)(2)(3) 三小題

Problem 7. Let R be the region bounded by the circle $r = 1$ and outside the lemniscate $r^2 = -2\cos 2\theta$, and is located on the right half plane (see the shaded region in the graph).

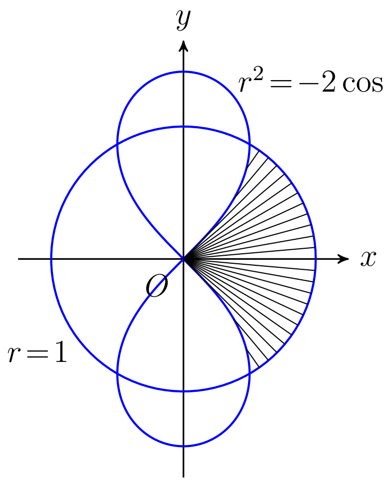


Figure 2: The polar graphs of the polar equations $r = 1$ and $r^2 = -2\cos 2\theta$

(1) Find the points of intersection of the circle $r = 1$ and the lemniscate $r^2 = -2\cos 2\theta$.

(2) Show that the straight line $x = \frac{1}{2}$ is tangent to the lemniscate at the points of intersection on the right half plane.

(3) Find the area of R.

(4) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.4}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.4), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1 - 4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = \pi \int_0^{\frac{1}{2}} c(x)dx + \pi \int_{\frac{1}{2}}^1 d(x)dx.$$

Compute $\displaystyle\int_{\frac{1}{2}}^{1} \Big[d(x) - (1-x^2)\Big]dx.$

(c) Evaluate I by first computing the integral $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-4x^2}dx$, and then find I.

(5) Find the area of the surface of revolution obtained by rotating the boundary of R about the x-axis.

Problem 8. Let R be the region bounded by the lemniscate $r^2 = 2\cos 2\theta$ and is outside the circle $r = 1$ (see the shaded region in the graph).

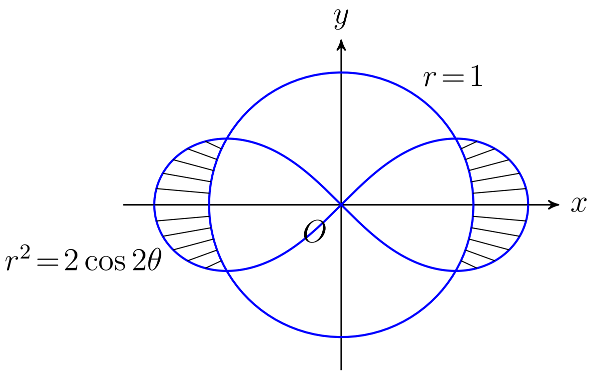


Figure 3: The polar graphs of the polar equations $r^2 = 2\cos 2\theta$ and $r = 1$

(1) Find the area of R.

(2) Find the slope of the tangent line passing thought the point on the lemniscate corresponding to $\theta = \dfrac{\pi}{6}$.

(3) Find the volume of the solid of revolution obtained by rotating R about the x-axis by complete the following:

(a) Suppose that (x, y) is on the lemniscate. Then (x, y) satisfies

$$y^4 + a(x)y^2 + b(x) = 0 \tag{0.5}$$

for some functions $a(x)$ and $b(x)$. Find $a(x)$ and $b(x)$.

(b) Solving (0.5), we find that $y^2 = c(x)$, where $c(x) = c_1x^2 + c_2 + c_3\sqrt{1+4x^2}$ for some constants c_1, c_2 and c_3. Then the volume of interests can be computed by

$$I = 2 \times \Big[\pi \int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} c(x)dx - \pi \int_{\frac{\sqrt{3}}{2}}^{1} d(x)dx\Big].$$

Compute $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{1} \big[d(x) - (1-x^2)\big]dx.$

(c) Evaluate I by first computing the integral $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{\sqrt{2}} \sqrt{1+4x^2}dx$, and then find I.

(4) Find the surface area of the surface of revolution obtained by rotating the boundary of R about the x-axis.

Problem 9. Let C_1, C_2 be the curves given by polar coordinate $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$, respectively, and the graphs of C_1 and C_2 are given in Figure 4.

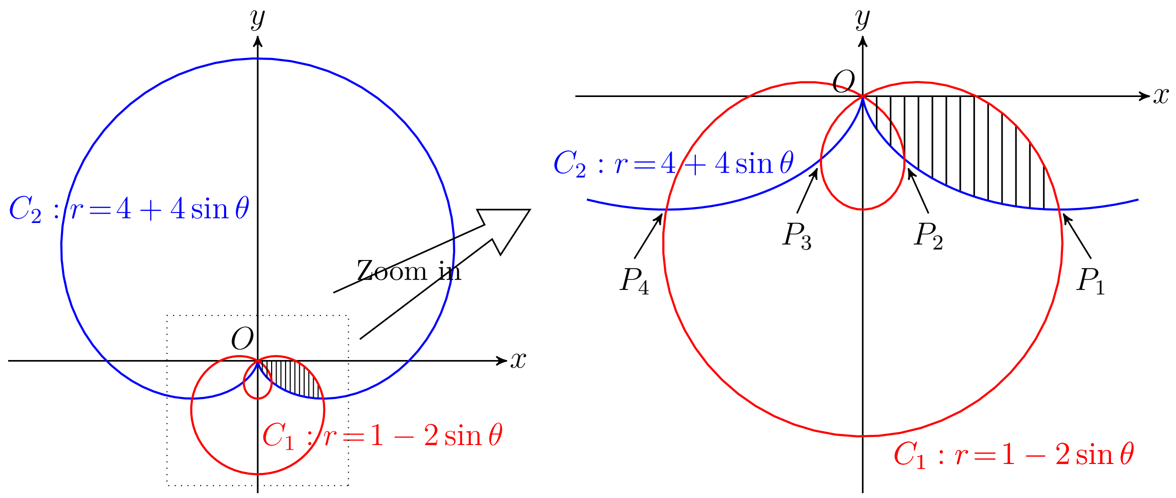


Figure 4: The polar graphs of the polar equations $r = 1 - 2\sin\theta$ and $r = 4 + 4\sin\theta$

(1) Let P_1, $\cdots$, P_4 be four points of intersection of curves C_1 and C_2 as shown in Figure 4 (the fifth one is the origin). What are the Cartesian coordinates of P_1 and P_2?

(2) Let L_1 and L_2 be two straight lines passing P_1 and tangent to C_1, C_2, respectively. Find the cosine value of the acute/smaller angle between L_1 and L_2.

(3) Compute the area of the shaded region.