Exercise Problem Sets 5

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Problem 1. Let $f : (-r, r) \to \mathbb{R}$ be n-times differentiable at 0, and $P_n(x)$ be the n-th Maclaurin polynomial for f.

1. Show that if $g(x) = x^\ell f(x^m)$ for some positive integers m and ℓ, then the $(mn+\ell)$-th Maclaurin polynomial for g is $x^\ell P_n(x^m)$.

2. Show that if $h(x) = x^\ell f(-x^m)$ for some positive integers m and ℓ, then the $(mn+\ell)$-th Maclaurin polynomial for h is $x^\ell P_n(-x^m)$.

3. Find the Maclaurin series for the following functions:

   (1) $y = \dfrac{1}{1+x^2}$ (2) $y = x^2 \arctan(x^3)$ (3) $y = \ln(1+x^4)$ (4) $y = x\sin(x^3)\cos(x^3)$.

Hint for (1) and (2): See Exercise 3 Problem 4.

Problem 2. To find the sum of the series $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}$, express $\dfrac{1}{1-x}$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to $\dfrac{1}{2}$. What do you get?

Problem 3. Complete the following.

(1) Use the power series of $y = \arctan x$ to show that

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

(2) Using $x^3 + 1 = (x+1)(x^2 - x + 1)$, rewrite the integral $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x^2 - x + 1}$ and then express $\dfrac{1}{1+x^3}$ as the sum of a power series to prove the following formula for π:

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \left(\frac{2}{3n+1} + \frac{1}{3n+2} \right).$$

Problem 4. Show that the Bessel function of the first kind of order 0, denoted by J_0 and defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2},$$

satisfies the differential equation

$$x^2 y''(x) + x y'(x) + x^2 y(x) = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

Problem 5. Show that the Bessel function of the first kind of order 1, denoted by J_1 and defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}},$$

satisfies the differential equation

$$x^2 y''(x) + xy'(x) + (x^2 - 1)y(x) = 0, \qquad y(0) = 0,\ y'(0) = \frac{1}{2}.$$

Problem 6. Suppose that $x_1(t)$ and $x_2(t)$ are functions of t satisfying the following equations

$$\begin{aligned} x_1''(t) - x_1(t) &= 0, \qquad & x_1(0) &= 1, \quad & x_1'(0) &= 0, \\ x_2''(t) - x_2(t) &= 0, \qquad & x_2(0) &= 0, \quad & x_2'(0) &= 1, \end{aligned}$$

where $'$ denotes the derivatives with respect to t.

1. Assume that the function $x_1(t)$ and $x_2(t)$ can be written as a power series (on a certain interval), that is, $x_1(t) = \sum\limits_{k=0}^{\infty} a_k t^k$ and $x_2(t) = \sum\limits_{k=0}^{\infty} b_k t^k$. Show that

$$(k+2)(k+1)a_{k+2} = a_k \ \text{ and } \ (k+2)(k+1)b_{k+2} = b_k \qquad \forall\, k \geqslant 0.$$

2. Find a_k and b_k, and conclude that x_1 and x_2 are some functions that we have seen before.

3. Find a function $x(t)$ satisfying

$$x''(t) - x(t) = 0, \qquad x(0) = a, \quad x'(0) = b.$$

Note that x can be written as the linear combination of x_1 and x_2.

Problem 7. Find the series solution to the differential equation

$$y''(x) + x^2 y(x) = 0, \qquad y(0) = 1\, y'(0) = 0.$$

What is the radius of convergence of this series solution?

Problem 8. In this problem we try to establish the following theorem

> Let the radius of convergence of the power series $f(x) = \sum\limits_{k=0}^{\infty} a_k (x-c)^k$ be r for some $r > 0$.
>
> 1. If $\sum\limits_{k=0}^{\infty} a_k r^k$ converges, then f is continuous at $c+r$; that is, $\lim\limits_{x \to (c+r)^-} f(x) = f(c+r)$.
> 2. If $\sum\limits_{k=0}^{\infty} a_k (-r)^k$ converges, then f is continuous at $c-r$; that is, $\lim\limits_{x \to (c-r)^+} f(x) = f(c-r)$.

Prove case 1 of the theorem above through the following steps.

(1) Let $A = \sum_{k=0}^{\infty} a_k r^k$, and define

$$g(x) = f(rx + c) - A = -\sum_{k=1}^{\infty} a_k r^k + \sum_{k=1}^{\infty} a_k r^k x^k = \sum_{k=0}^{\infty} b_k x^k \,,$$

where $b_k = a_k r^k$ for each $k \in \mathbb{N}$ and $b_0 = -\sum_{k=1}^{\infty} a_k r^k$. Show that the radius of convergence of g is 1 and $\sum_{k=0}^{\infty} b_k = 0$. Moreover, show that f is continuous at $c + r$ if and only if g is continuous at 1.

(2) Let $s_n = b_0 + b_1 + \cdots + b_n$ and $S_n(x) = b_0 + b_1 x + \cdots + b_n x^n$. Show that

$$S_n(x) = (1 - x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n$$

and conclude that

$$g(x) = \lim_{n\to\infty} S_n(x) = (1 - x) \sum_{k=0}^{\infty} s_k x^k \,. \tag{$\star$}$$

(3) Use ($\star$) to show that g is continuous at 1. Note that you might need to use ε-δ argument.