Exercise Problem Sets 4

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Problem 1. Suppose that $f : [a,b] \to \mathbb{R}$ is three times continuously differentiable, $h = \dfrac{b-a}{2}$ and $c = \dfrac{a+b}{2}$. Show that there exists $\xi \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{2h} - \frac{h^2}{6} f^{(3)}(\xi)\,.$$

Hint: Find the difference $f(b) - f(a)$ by write f as the sum of its third Taylor polynomial about c and the corresponding remainder. Apply the Intermediate Value Theorem to deal with the sum of the remainders. We note that the identity above implies that

$$\Big| f'(c) - \frac{f(c+h)-f(c-h)}{2h} \Big| \leqslant \frac{h^2}{6} \max_{x\in[c-h,c+h]} \big| f^{(3)}(x) \big|\,.$$

Problem 2. Suppose that $f : [a,b] \to \mathbb{R}$ is four times continuously differentiable, $h = \dfrac{b-a}{2}$ and $c = \dfrac{a+b}{2}$. Show that there exists $\xi \in (a,b)$ such that

$$f''(c) = \frac{f(a) - 2f(c) + f(b)}{h^2} - \frac{f^{(4)}(\xi)}{12} h^2\,. \tag{$\star$}$$

Hint: Find the sum $f(a) + f(b)$ by write f as the sum of its third Taylor polynomial about c and the corresponding remainder. Apply the Intermediate Value Theorem to deal with the sum of the remainders. We note that the identity above implies that

$$\Big| f''(c) - \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} \Big| \leqslant \frac{h^2}{12} \max_{x\in[c-h,c+h]} \big| f^{(4)}(x) \big|\,.$$

Problem 3. Suppose that $f : [a,b] \to \mathbb{R}$ is four times continuously differentiable. Show that

$$\Big| \int_a^b f(x)\,dx - \frac{b-a}{6}\big[f(a) + 4f\big(\frac{a+b}{2}\big) + f(b)\big] \Big| \leqslant \frac{2h^5}{45} \max_{x\in[a,b]} \big| f^{(4)}(x) \big| \tag{$\diamond$}$$

through the following steps.

1. Let $c = \dfrac{a+b}{2}$ and $h = \dfrac{b-a}{2}$. Write f as the sum of its third Taylor polynomial about c and the corresponding remainder and conclude that

$$\int_a^b f(x)\,dx = 2hf(c) + \frac{h^3}{3} f''(c) + \int_a^b R_3(x)\,dx\,.$$

2. Show (by Intermediate Value Theorem) that there exists $\xi \in (a,b)$ such that

$$\int_a^b R_3(x)\,dx = \frac{f^{(4)}(\xi)}{24} \int_a^b (x-c)^4\,dx = \frac{f^{(4)}(\xi)}{60} h^5\,. \tag{$\star\star$}$$

3. Use ($\star$) in ($\star\star$) and conclude ($\diamond$).

Problem 4. Find the interval of convergence of the following power series.

(1) $\sum\limits_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n}x^{n}$ (2) $\sum\limits_{n=1}^{\infty}(\ln n)x^{n}$ (3) $\sum\limits_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)x^{n}$ (4) $\sum\limits_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}x^{n}$

(5) $\sum\limits_{n=1}^{\infty}\frac{n!}{(2n)!}x^{n}$ (6) $\sum\limits_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}x^{2n+1}$ (7) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n}3\cdot 7\cdot 11\cdot\cdots\cdot(4n-1)}{4^{n}}(x-3)^{n}$

(8) $\sum\limits_{n=1}^{\infty}\frac{1}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}x^{n}$ (10) $\sum\limits_{n=1}^{\infty}\frac{1}{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}x^{n}$ (9) $\sum\limits_{n=1}^{\infty}\frac{n!}{3\cdot 6\cdot 9\cdot\cdots\cdot(3n)}x^{n}$

(10) $\sum\limits_{n=1}^{\infty}\frac{k(k+1)(k+2)\cdots(k+n-1)}{n!}x^{n}$, where k is a positive integer;

(11) $\sum\limits_{n=0}^{\infty}\frac{(n!)^{k}}{(kn)!}x^{n}$, where k is a positive integer; (12) $\sum\limits_{n=2}^{\infty}\frac{x^{n}}{n\ln n}$ (13) $\sum\limits_{n=2}^{\infty}\frac{x^{n}}{n(\ln n)^{2}}$

(14) $\sum\limits_{n=1}^{\infty}\left[2+(-1)^{n}\right](x+1)^{n-1}$

Problem 5. The function J_0 defined by

$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n}}{2^{2n}(n!)^{2}}$$

is called the Bessel function of the first kind of order 0. Find its domain (that is, the interval of convergence).

Problem 6. The function J_1 defined by

$$J_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n}x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

is called the Bessel function of the first kind of order 1. Find its domain (that is, the interval of convergence).

Problem 7. The function A defined by

$$A(x)=1+\frac{x^{3}}{2\cdot 3}+\frac{x^{6}}{2\cdot 3\cdot 5\cdot 6}+\frac{x^{9}}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\cdots$$

is called an Airy function after the English mathematician and astronomer Sir George Airy (1801–1892). Find the domain of the Airy function.

Problem 8. A function f is defined by

$$f(x)=1+2x+x^{2}+2x^{3}+x^{4}+\cdots;$$

that is, its coefficients are $c_{2n}=1$ and $c_{2n+1}=2$ for all $n\geqslant 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.