Exercise Problem Sets 3

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Problem 1. The second Taylor polynomial for a twice-differentiable function f at $x = c$ is called the quadratic approximation of f at $x = c$. Find the quadratic approximate of the following functions at $x = 0$.

(1) $f(x) = \ln \cos x$ (2) $f(x) = e^{\sin x}$ (3) $f(x) = \tan x$ (4) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

(5) $f(x) = e^x \sin^2 x$ (6) $f(x) = e^x \ln(1+x)$ (7) $f(x) = (\arctan x)^2$

Problem 2. Let f have derivatives through order n at $x = c$. Show that the n-th Taylor polynomial for f at c and its first n derivatives have the same values that f and its first n derivatives have at $x = c$.

Problem 3. Complete the following.

(1) Let $f, g : [a, b] \to \mathbb{R}$ be continuous and g is sign-definite; that is, $g(x) \geqslant 0$ for all $x \in [a, b]$ or $g(x) \leqslant 0$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that

$$f(c) \int_a^b g(x)\, dx = \int_a^b f(x) g(x)\, dx\,. \qquad (\star)$$

(2) Let $f : [a, b] \to \mathbb{R}$ be a function, and $c \in [a, b]$. Prove (by induction) that if f is $(n+1)$-times continuously differentiable on $[a, b]$, then for all $x \in [a, b]$,

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &\quad + (-1)^n \int_c^x f^{(n+1)}(t) \frac{(t-x)^n}{n!}\, dt \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n \int_c^x f^{(n+1)}(t) \frac{(t-x)^n}{n!}\, dt\,. \end{aligned}$$

(3) Use $(\star)$ to show that if f is $(n+1)$-times continuously differentiable on $[a, b]$ and $c \in [a, b]$, then for all $x \in [a, b]$ there exists a point ξ between x and c such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\,.$$

(4) Find and explain the difference between the conclusion above and Taylor's Theorem.

Problem 4. Suppose that f is differentiable on an interval centered at $x = c$ and that $g(x) = b_0 + b_1(x-c) + \cdots + b_n(x-c)^n$ is a polynomial of degree n with constant coefficients b_0, b_1, $\cdots$, b_n. Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

1. $E(c) = 0$ (which means "the approximation error is zero at $x = c$");

2. $\lim_{x \to c} \frac{E(x)}{(x-c)^n} = 0$ (which means "the error is negligible when compared to $(x-c)^n$),

then g is the n-th Taylor polynomial for f at c. **Thus, the Taylor polynomial P_n is the only polynomial of degree less than or equal to n whose error is both zero at $x = c$ and negligible when compared with $(x-c)^n$.**

Problem 5. Show that if p is an polynomial of degree n, then

$$p(x+1) = \sum_{k=0}^{n} \frac{p^{(k)}(x)}{k!} \,.$$

Problem 6. In Chapter 3 we considered Newton's method for approximating a root/zero r of the equation $f(x) = 0$, and from an initial approximation x_1 we obtained successive approximations x_2, x_3, $\cdots$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \forall\, n \geqslant 1 \,.$$

Show that if f'' exists on an interval I containing r, x_n, and x_{n+1}, and $\left|f''(x)\right| \leqslant M$ and $\left|f'(x)\right| \geqslant K$ for all $x \in I$, then

$$|x_{n+1} - r| \leqslant \frac{M}{2K} |x_n - r|^2$$

This means that if x_n is accurate to d decimal places, then x_{n+1} is accurate to about $2d$ decimal places. More precisely, if the error at stage n is at most 10^{-m}, then the error at stage $n+1$ is at most $\frac{M}{2K} 10^{-2m}$.

Hint: Apply Taylor's Theorem to write $f(r) = P_2(r) + R_2(r)$, where P_2 is the second Taylor polynomial for f at x_n.

Problem 7. Consider a function f with continuous first and second derivatives at $x = c$. Prove that if f has a relative maximum at $x = c$, then the second Taylor polynomial centered at $x = c$ also has a relative maximum at $x = c$.