Exercise Problem Sets 2

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Problem 1. Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

(1) $a_n = \dfrac{1}{n^{1+\frac{1}{n}}}$ (2) $a_n = \ln\big(1+\dfrac{1}{n^2}\big)$ (3) $a_n = \dfrac{2^n+3^n}{3^n+4^n}$ (4) $a_n = \tan\dfrac{1}{n}$

(5) $a_n = \sin^n \dfrac{1}{\sqrt{n}}$ (6) $a_n = \dfrac{\arctan n}{n^{1.1}}$ (7) $a_n = \Big[-\ln\big(e^2+\dfrac{1}{n^2}\big)\Big]^n$ (8) $a_n = \big(1-\dfrac{1}{n}\big)^{n^2}$

(9) $a_n = \big(1+\dfrac{1}{n}\big)^{-n^2}$ (10) $a_n = \dfrac{(n!)^2}{(2n)!}$ (11) $a_n = \dfrac{n!\ln n}{n(n+2)!}$ (12) $a_n = \dfrac{n!}{n^n}$

(13) $a_n = \dfrac{(-1)^n(3n)!}{n!(n+1)!(n+2)!}$ (14) $a_n = \dfrac{(-1)^n(n!)^n}{n^{n^2}}$ (15) $a_n = \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}$

(16) $a_n = (-1)^n\big(\sqrt{n+\sqrt{n}}-\sqrt{n}\big)$ (17) $a_n = (-1)^n\dfrac{(n!)^2 3^n}{(2n+1)!}$

Problem 2. Find all p and q such that $\sum_{k=2}^{\infty}\dfrac{(\ln k)^q}{k^p}$ converges.

Problem 3. Show that if $\sum_{k=1}^{\infty} a_k$ is a convergent series of positive terms, then $\sum_{k=1}^{\infty}\sin a_k$ converges.

Problem 4. Let $S = \sum_{k=1}^{\infty}\dfrac{1}{k^2}$. Euler found that $S = \dfrac{\pi^2}{6}$ in 1735 AD.

(1) Show that $S = 1 + \sum_{k=1}^{\infty}\dfrac{1}{n^2(n+1)}$.

(2) Which of the sums $\sum_{k=1}^{1000000}\dfrac{1}{k^2}$ or $1+\sum_{k=1}^{1000}\dfrac{1}{k^2(k+1)}$ should give a better approximation of S? Explain your answer.

Hint: (1) $\dfrac{1}{n^2(n+1)} = \dfrac{1}{n^2} - \dfrac{1}{n(n+1)}$.

Problem 5. Find all real numbers x such that $\sum_{k=1}^{\infty}\dfrac{\cos(kx)}{\ln k}$ converges.

Problem 6. Show by example that $\sum_{k=1}^{\infty} a_k b_k$ may diverge even if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge.

Problem 7. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n, b_n > 0$ for all $n \geqslant N$. Define

$$c_n = b_n - b_{n+1}\frac{a_{n+1}}{a_n} \qquad \forall\, n \in \mathbb{N}\,. \tag{$\star$}$$

1. Show that if there exists a constant $r > 0$ such that $r < c_n$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ converges.

   Hint: Rewrite ($\star$) as $b_n = c_n + \dfrac{a_{n+1}}{a_n} b_{n+1}$ and then obtain

$$\begin{aligned} b_N &= c_N + \frac{a_{N+1}}{a_N} b_{N+1} = c_N + \frac{a_{N+1}}{a_N}\Big(c_{N+1} + \frac{a_{N+2}}{a_{N+1}} b_{N+2}\Big) = c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} b_{N+2} \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N}\Big(c_{N+2} + \frac{a_{N+3}}{a_{N+2}} b_{N+3}\Big) = \cdots \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} c_{N+2} + \cdots + \frac{a_{N+n}}{a_N} c_{N+n} + \frac{a_{N+n+1}}{a_N} b_{N+n+1}\,. \end{aligned}$$

   Use the fact that $0 < r < c_n$ for all $n \geqslant N$ to conclude that

$$\sum_{k=N}^{N+n} a_k \leqslant \frac{a_N b_N}{r} \qquad \forall\, n \in \mathbb{N}\,.$$

   Note that then the sequence of partial sum of $\sum\limits_{k=1}^{\infty} a_k$ then is bounded from above $\Big($by $\sum\limits_{k=1}^{N-1} a_k + \dfrac{a_N b_N}{r}\Big)$.

2. Show that if $\sum\limits_{k=1}^{\infty} \dfrac{1}{b_k}$ diverges and $c_n \leqslant 0$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

   Hint: The fact that $c_n \leqslant 0$ for all $n \geqslant N$ implies that $b_n a_n \leqslant b_{n+1} a_{n+1}$ for all $n \geqslant N$. Use this fact to conclude that

$$\frac{a_N b_N}{b_n} \leqslant a_n \qquad \forall\, n \geqslant N$$

   and then apply the direct comparison test to conclude that $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Problem 8. Let $\sum\limits_{k=1}^{\infty} a_k$ be a series with positive terms, and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$. We know from class that the ratio test fails when this happens, but there are some refined results concerning this particular case.

1. **(Raabe's test)**:

   (a) If there exists a constant $\mu > 1$ such that $\dfrac{a_{n+1}}{a_n} < 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ converges.

   (b) If there exists a constant $0 < \mu < 1$ such that $\dfrac{a_{n+1}}{a_n} > 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

   Hint: Consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_n = (n-1)a_n - na_{n+1}$. Then $\sum\limits_{k=1}^{\infty} b_k$ is a telescoping series. For case (a), show that $\big\{na_{n+1}\big\}_{n=N}^{\infty}$ is a positive decreasing sequence and then conclude that $\sum\limits_{k=1}^{\infty} b_k$ converges. Note that $b_n \geqslant (\mu - 1)a_n$ for all $n \geqslant N$. For case (b), show that $\big\{na_{n+1}\big\}_{n=N}^{\infty}$ is a positive increasing sequence; thus $a_n \geqslant \dfrac{Na_{N+1}}{n-1}$ for all $n \geqslant N+1$ which implies that $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Remark: 注意到 (a) 說的是如果 $\{a_n\}_{n=1}^{\infty}$ 在某項之後「遞減得夠快」，那麼 $\sum\limits_{k=1}^{\infty} a_k$ 收斂。反之，如果 $\{a_n\}_{n=1}^{\infty}$「並非遞減得那麼快」，那麼 $\sum\limits_{k=1}^{\infty} a_k$ 發散。

2. **(Gauss's test)**: Suppose that there exist a positive constant $\epsilon > 0$, a constant μ, and a **bounded** sequence $\{R_n\}_{n=1}^{\infty}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} \qquad \text{for all } n \geqslant N\,.$$

(a) If $\mu > 1$, then $\sum\limits_{k=1}^{\infty} a_k$ converges. (b) If $\mu \leqslant 1$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Hint: Show that if $\mu > 1$ or $\mu < 1$, one can apply Raabe's test to conclude Gauss's test. For the case $\mu = 1$, let $b_n = (n-1)\ln(n-1)$ for $n \geqslant 2$. Using the second result of Problem 7 to show the divergence of $\sum\limits_{k=1}^{\infty} a_k$ (by showing that c_n defined by $(\star)$ is non-positive for all large enough n).

Problem 9. Complete the following.

1. Show that $\sum\limits_{k=1}^{\infty} \Big(1 - \dfrac{1}{\sqrt{k}}\Big)^k$ converges.
2. Show that $\sum\limits_{k=2}^{\infty} \dfrac{\log(k+1) - \log k}{(\log k)^2}$ converges.
3. Use Gauss's test to show that both the general harmonic series $\sum\limits_{k=1}^{\infty} \dfrac{1}{ak+b}$, where $a \neq 0$, and the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{\sqrt{k}}$ diverge.
4. Show that $\sum\limits_{k=1}^{\infty} \dfrac{k!}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}$ converges if $\alpha > 1$ and diverges if $\alpha \leqslant 1$.
5. Test the following "hypergeometric" series for convergence or divergence:

(a) $\sum\limits_{k=1}^{\infty} \dfrac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+k-1)} = \dfrac{\alpha}{\beta} + \dfrac{\alpha(\alpha+1)}{\beta(\beta+1)} + \dfrac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \cdots.$

(b) $1 + \dfrac{\alpha\cdot\beta}{1\cdot\gamma} + \dfrac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\gamma\cdot(\gamma+1)} + \dfrac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)} + \cdots.$

Problem 10. Let $\sum\limits_{k=1}^{\infty} a_k$ be a conditionally convergent series. Show that $\sum\limits_{k=1}^{\infty} \big[1 + \mathrm{sgn}(a_k)\big] a_k$ and $\sum\limits_{k=1}^{\infty} \big[1 - \mathrm{sgn}(a_k)\big] a_k$ both diverge. Here the sign function sgn is defined by

$$\mathrm{sgn}(a) = \begin{cases} 1 & \text{if } a > 0\,, \\ 0 & \text{if } a = 0\,, \\ -1 & \text{if } a < 0\,. \end{cases}$$

Problem 11. A permutation of a non-empty set A is a one-to-one function from A onto A. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$.

1. Suppose that $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers. Show that $\left\{a_{\pi(n)}\right\}_{n=1}^{\infty}$ is also convergent; that is, show that if $\{b_n\}_{n=1}^{\infty}$ is a sequence defined by $b_n = a_{\pi(n)}$, then $\{b_n\}_{n=1}^{\infty}$ also converges.

2. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum\limits_{k=1}^{\infty} a_{\pi(k)}$ is also absolutely convergent, and
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\pi(k)} \,.$$

3. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is conditionally convergent. Show that for each $r \in \mathbb{R}$, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that
$$\sum_{k=1}^{\infty} a_{\pi(k)} = r \,.$$